GRADUATED EXERCISES

AND

EXAMPLES

FOR THE USE OF STUDENTS

OF THE

INSTITUTE OF ACTUARIES' TEXT-BOOK.

PART I.—INTEREST (including Annuities-Certain).

PART II.—LIFE CONTINGENCIES (including Life Annuities and Assurances).

WITH SOLUTIONS.

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INTRODUCTION.

THE following EXERCISES AND EXAMPLES are intended to supply a systematic course of work for the use of Actuarial Students, in practical illustration and application of the principles and formulas laid down and developed in the INSTITUTE OF ACTUARIES' TEXT-BOOK, PARTS I. and II.

The Examples upon *Interest and Annuities-Certain* have been carefully selected to supplement those printed at the end of the INSTITUTE OF ACTUARIES' TEXT-BOOK, PART I., and in further illustration of the text and demonstrations of that work.

The Examples in *Life Contingencies* were originally intended to be appended to the INSTITUTE OF ACTUARIES' TEXT-BOOK, PART II., but their insertion was ultimately found to be impracticable, owing to the already considerable bulk of that treatise. It was also felt that the usefulness of such Examples would be greatly increased if they were accompanied by short Solutions and references to the Text-Book.

The present Volume has accordingly been undertaken, and is now issued upon the sole responsibility of the joint Authors, who have selected the Exercises and prepared the Solutions here given, in the light of their practical experience during the past six sessions (1883-4 to 1888-9 inclusive) as successive Lecturers to Students of the Institute of Actuaries upon the subjects of the Part II. Examination.

The Exercises and Examples have been selected from all available sources in Actuarial literature, and largely from the Examination Papers of past years. In a few cases, where a practical illustration of some important theorem or demonstration treated in the Text-Book was not readily available, special Examples have been prepared to meet the case.

In so considerable a body of work it is hardly probable that errors will have been entirely eliminated. The greatest care has, however, been taken to secure accuracy; and the Compilers desire to express their special indebtedness to Mr. WILLIAM SMITH ANDERSON, A.I.A., and Mr. HARRY BEARMAN, A.I.A., both of the GRESHAM LIFE ASSURANCE SOCIETY, for valued assistance in the careful examination of the Exercises and Solutions, and in the revision of proof-sheets.

T. G. A.
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NOTES FOR THE STUDENT.

THE EXERCISES AND EXAMPLES are arranged in two main divisions—*Interest* and *Life Contingencies*—corresponding respectively with Parts I. and II. of the INSTITUTE OF ACTUARIES' TEXT-BOOK. The chapters into which the Exercises are divided correspond throughout with those similarly numbered in the Text-Book; and the Exercises contained in each chapter are arranged, as far as practicable, in the order in which the subjects are treated and developed in the corresponding chapter in the Text-Book. At the foot of each page of the Exercises and Examples is appended a reference to the pages, later on in the present volume, upon which the Solutions of such Exercises will be found.

The references, in the Solutions, to figures in square brackets—thus, § [16]—indicate the numbered paragraphs in the Text-Book in which the subject dealt with in the particular Example is treated; and the paragraphs so referred to will be found (unless otherwise stated) in the chapter of the Text-Book then under discussion. Any additional hints by way of solution, or references to other authorities, are appended for the guidance of the Student, and the further elucidation of the points raised in the several Exercises and Examples.

References to the *Journal of the Institute of Actuaries* are indicated by the letters *J.I.A.*

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GRADUATED EXERCISES AND EXAMPLES.

PART I.—INTEREST (INCLUDING ANNUITIES-CERTAIN).

CHAPTER I.—INTEREST, AMOUNTS, PRESENT VALUES, AND DISCOUNT.

(1).—Give the two formulas which express interest for a fractional period, and state what assumptions are made in each case, and the advantages and disadvantages of each formula.

(2).—Write down the formula for the amount of interest accrued on a sum at the end of the $\frac{m-1}{m}$th of a year, at the effective rate of i per annum, and prove that such accrued interest is less than the simple interest for the same period at the rate i? How would you account for this?

(3).—State clearly the meaning of the term "force of interest", and obtain a formula exhibiting its relation to the effective rate of interest.

(4).—Show that the instantaneous rate equivalent to a yearly rate of i is approximately equal to the arithmetic mean between the theoretical and commercial discount on 1 for a year, the rate of interest being i.

(5).—Write down the present value of a sum due 3, 6, and 9 months hence. State your reasons for the formulas adopted.

(6).—Given $\log_e 10 = 2{\cdot}3026$, show that 1 will amount to 10, at 1 per-cent, in $231\frac{2}{5}$ years.

(7).—Write down expressions for the present value and amount of 1 at the end of n years, interest being reckoned at the nominal rate of i, convertible m times a year. What do these expressions respectively become when $m = \infty$?

(8).—Given tables of the functions $(1{\cdot}0125)^n$ and $(1{\cdot}0125)^{-n}$ for integral values of n, how would you employ them to ascertain (α) the amount of £100 in 10 years, interest being reckoned at the rate of 5 per-cent per annum, payable quarterly; (β) the present value of £100 due 15 years hence, interest being reckoned at the rate of $2\frac{1}{2}$ per-cent per annum, payable half-yearly?

(9).—State symbolically the difference between discount at simple interest, discount at compound interest, and commercial discount.

(10).—Show that $\delta = -\frac{1}{v^x}\frac{dv^x}{dx}$.

(11).—A gives B a bill for £a due at the end of m years, in discharge of a bill for £b due at the end of n years. For what sum should B give A a bill due at the end of p years to balance the account?

CHAPTER II.—ANNUITIES-CERTAIN.

(i) *Interest convertible yearly, and annuity payable yearly.*

(12).—Find the relation between the discount of any sum payable n years hence, and the present value of an annuity-certain for n years of the same amount.

(13).—An annuity-certain of £729 a year is granted for 25 years, the rate of interest at 5 per-cent. Calculate the value of such annuity, given, $\log 2 = {\cdot}30103$; $\log 7 = {\cdot}84510$; $\log 3 = {\cdot}47712$; $\log 2952 = 3{\cdot}47012$; $\log 2953 = 3{\cdot}47026$.

(14).—What sum would have to be deducted from the first payment of the annuity in the preceding question, if the first payment is made three months hence, the second in fifteen months, &c.?

(15).—Show how the several payments of an annuity-certain may be divided into principal and interest, and demonstrate that the amount remaining outstanding at the end of any year is equal to the present value of an annuity-certain for the remaining term.

(16).—Show that the amount of purchase-money of an annuity-certain for n years which is unpaid at the end of m years, is equal to the difference between the accumulated amount of the purchase-money for *m* years, and the accumulated amount of the annuity in the same time.

(17).—An insurance company lends £50,000, repayable by an annuity-certain for 25 years. How much capital is unpaid at the end of 20 years, reckoning interest at 5 per-cent?

(18).—In the above case construct a table showing how much of each annual instalment of annuity consists of interest, and how much of repayment of capital.

(19).—Show how to find the amount of each payment of an annuity-certain for *n* years, which is purchased for a sum of V, the purchaser wishing to invest his capital at a rate of interest i', and to replace that capital at a rate of interest i.

(20).—Show clearly into what component parts an annuity-certain may be divided, and give the means of determining how much of each payment goes to repay the capital originally invested, in the case where the rate of interest at which that part of each payment which goes to repay capital is invested, is different from the rate of interest realized on the original investment.

(21).—Find the present value of a perpetuity of 1, and hence deduce the present value of an annuity-certain for *n* years.

(22).—A lease of the annual value of 1 is granted for *m* years. After it has been *n* years in force, the lessee requires to extend the remaining term to *m* years. How much ought he to pay?

(23).—Given the present value, at rate of interest *i*, of an annuity of 1 deferred *d* years, investigate a formula to find *n*, the number of years it has to run.

(24).—Four persons, A, B, C, and D, contribute equal sums towards the purchase of a perpetuity. Find the number of years that A, B, and C may successively enjoy it, D having the absolute reversion, so that all four may benefit equally.

(ii) *Annuity payable and interest convertible at the same period, the period being less than a year.*

(25).—Find the present value and amount of an annuity-certain for *n* years, at a nominal rate of interest of *i*, convertible *m* times a year.

(26).—From the above, deduce the present values and amounts of continuous annuities-certain. Show how a table of the values of $\frac{e^x-1}{x}$, x being the argument, enables us to calculate such values.

(27).—If two annuities-certain of 1 per annum for n years are purchased, in one of which the annuity is payable and interest convertible yearly, while in the other the instalment is payable and interest convertible m times a year, the effective rate of interest being the same in the two cases, what is the relation between the amount of capital repaid in the tth year under the two annuities respectively?

(28).—Give a formula expressing the value of an annuity-certain of 1 for n years, payable m times a year, in terms of the value of an annuity-certain of 1 for n years payable yearly, the nominal and the effective rates of interest. What does the formula become when $m=\infty$?

(29).—If x be the nominal rate of interest, the value of a perpetuity is equal to $\frac{1}{x}$, however often interest is convertible. Under what condition is this true, and how would you explain it?

(30).—Find the value of an annuity to run for five years, interest and instalment payable half-yearly, with interest at the nominal rate of 5 per-cent, and show the amount of capital redeemed in each half-yearly payment.

(31).—The guardians of a poor-law union are desirous of borrowing £5,000, the loan to be discharged and the interest paid by an equal half-yearly charge upon the rates, extending over 30 years. Assuming 5 per-cent interest, find the amount of the half-yearly charge, and draw up a schedule showing what portions of each of the first four payments are applicable for the payment of interest and the discharge of the capital account.

(iii) *Annuity payable and interest convertible at periods of different duration.*

(32).—Write down the formulas for the present value and amount of an annuity-certain for n years, instalment payable k, and interest m times a year, in terms of the effective rate of interest i and also of the nominal rate of interest x.

(33).—Given the formula for the present value of an annuity-certain for n years, instalment payable k, and interest m times a year, show how the expression will be modified in the several cases where $m=1$,

$k=1$, $m=k=1$, $m=\infty$, $k=\infty$, $m=k=\infty$, $m=1$, $k=\infty$, $k=1$, $m=\infty$.

(34).—Under what differing conditions are the following formulas, (α), (β), for the present value of a continuous annuity-certain for n years, correctly stated? What does the formula (γ) represent?

$$(\alpha)\ \frac{1-\epsilon^{-\delta n}}{\delta}, \qquad (\beta)\ \frac{1-\epsilon^{-in}}{i}, \qquad (\gamma)\ \frac{1-\epsilon^{-\delta n}}{i}.$$

(35)—A loan of X is to be discharged by an annuity (maae up of principal and interest) of $\frac{X}{10}$, payable at the end of each year, the interest thereon being at i per unit per annum, convertible half-yearly. When will the debt be extinguished?

CHAPTER III.—VARYING ANNUITIES.

(36).—Show that $a_{\overline{n}|\overline{r}|}=\left(\frac{v}{1-v}\right)^{r}$, stopping at the term involving v^n.

(37).—Find the present value of an annuity-certain, the first payment being 1, the second 1·2, the third 1·4, and so on, increasing ·2 each year; the last payment being 10.

(38).—Find the value of an annuity-certain for 21 years, the first payment of which is 1, and the after-payments of which increase by $\frac{1}{20}$th each.

(39).—Find the value of an annuitv-certain for n years, the payments of which are 1^3, 2^3, 3^3, &c.

(40).—State a gencral formula for the value of an annuity-certain for n years, whose successive payments are u_1, u_2, $u_3 \ldots u_n$; and explain how the values of annuities of the figurate numbers can be employed in determining such values.

(41).—Deduce a formula for the present value of a perpetuity whose successive payments are the figurate numbers of the rth order.

(42).—Find, by the method indicated by Mr. Peter Gray (*J.I.A.*, xiv, 91), the value of an annuity to run for 40 years, the successive payments of which are 1, 3, 5, 13, 33, . . ., interest being reckoned at 5 per-cent.

(43).—Find the value, at 5 per-cent, of an annuity for 40 years, whose several payments are 55, 126, 259, 484, 837, . . .

(44).—Find the value of an annuity for 15 years, whose successive payments are 201, 303, 443, 630, 874, . . .

CHAPTER IV.

(i) *On the determination of the rate of interest where the amount of capital repaid is the same as that advanced.*

(45).—An annuity-certain for 100 years is worth 19 years' purchase; find the rate of interest by formulas (A), (B), (C), (D).

(46).—An annuity-certain for 25 years is worth 17 years' purchase; find the rate of interest by formulas (D_1) and (D_2).

(47).—Show how to approximate to the rate of interest in an annuity-certain in cases where the term is long, and hence deduce the rate of interest when $a_{\overline{94}|}=27$.

48.—Give at least three formulas for approximating to the rate of interest in an annuity-certain, and state which you would select as likely to give the most satisfactory result without excessive labour.

(ii) *On the determination of the actual rate of interest paid by a borrower, where the amount of capital repaid is different from the amount advanced.*

(49).—Deduce Makeham's formula for the value, at rate of interest i', of a loan bearing interest at rate i, and explain it verbally. What are the advantages of the use of this formula?

(50).—A loan of £s is repayable at par in n years, and bears interest meanwhile at the rate i. Deduce a formula for the amount that could be given by a purchaser in order that interest at the rate of j may be realized upon the amount invested.

(51).—Obtain a formula for the present value of the capital redeemed in the successive payments of an annuity-certain of 1 for n years.

(52).—A loan is repaid by a sinking fund of p per-cent on the sum borrowed: find how long it will take to repay the loan, i and i' being the rates of interest which the borrower pays, and at which the sinking fund is invested, respectively.

(53).—A debenture, redeemable in n years, and bearing interest at a fixed rate i, is purchased at a premium of p per-cent. Show how to find approximately the rate of interest i', realized by the purchaser

(54).—If a bond of 1, repayable at par in n years, and bearing interest at the rate i, be purchased at the price of $1+p$, so that the purchaser may realize interest at the rate i' upon his investment, with the return of the capital invested at the end of the term, show that the excess interest received annually $[i-(1+p)i']$ must be invested at the rate i' to amount to p at the end of the n years. If this excess interest can only be invested at the rate j, how will the value of p, the premium to be paid for the bond, be affected?

(55).—A bond for £1,000, bearing interest at 3 per-cent for 20 years, is to be sold. What can a purchaser give to realize 5 per-cent from his investment (α) supposing the bond to be repayable at par in 20 years, (β) supposing the bond to be repayable in 20 annual instalments. Given $a_{\overline{20}|}$ at 5 per-cent=12·4622.

(56).—The tenant in possession of an estate has to pay off a charge on the estate in the next n years, in the following way, viz.: M being the amount of the charge, he has to pay back each year $\frac{M}{n}$, and interest at the rate of i on the amount unpaid at the beginning of the year. Find the sum for which his payments might be commuted, taking interest at the rate j.

(57).—A person spends in the first year m times the interest on his property; in the second year, $2m$ times; in the third year, $3m$ times, and so on; and at the end of $2p$ years has nothing left. Show that in the pth year he spends as much as he had left at the end of that year.

(58).—Find the rate of interest in a loan issued at 76 per-cent, with interest at 6 per-cent upon the nominal amount of the loan, and repayment of the nominal capital by an accumulative sinking fund of 1 per-cent per annum.

CHAPTER VII.—INTEREST TABLES.

(59).—Explain the methods of calculating and verifying tables of present values of sums and annuities-certain.

(60).—Calculate, at 5 per-cent, the present value of 1 due in any number of years from 1 to 20, and also the present values of annuities-certain for any number of years from 1 to 20, verifying the calculation in each case.

(61).—Show how tables where the argument is log x and the tabular results $\log(1-x)$ and $\log(1+x)$ can be made available for the calculation of the amounts and present values of annuities-certain.

(62).—Calculate by the tables referred to in the previous example the present values and amounts of annuities-certain for any number of years from 1 to 10, taking 5 as the rate of interest.

(63).—State and prove the rules for finding from the tabular values and amounts of annuities-certain, the values and amounts of the same when payable in advance.

(64).—How would you proceed to verify the columns $(1+i)^n$, v^n, $s_{\overline{n}|}$, and $a_{\overline{n}|}$, in a printed table of such values?

MISCELLANEOUS EXAMPLES.

(65).—A £100 share, bearing dividends at 5 per-cent per annum in June and December, with an annual bonus of £2 in December, is bought for £130, just after the payment of the December dividend and bonus. What is the effective rate of interest made upon the investment?

(66).—In the previous question, what would be the equivalent price just previous to the June dividend?

(67).—An assurance fund at the commencement of a year amounts to £1,000,000; the income from interest is £45,000, from premiums and other sources £200,000, the outgo £170,000. What is the rate of interest earned by the fund

(α) assuming the fund to increase at a uniform rate throughout the year;

(β) assuming the income (excluding interest), and the outgo, to be evenly distributed through the year?

(68).—If £100 amounts to 106·1678 in a year, at a nominal rate of interest of 6 per-cent: required to find how often interest must be convertible.

(69).—If $2\frac{1}{2}$ per-cent Consols are bought for 94, what are the nominal, effective, and instantaneous rates of interest?

(70).—Show that the present value of an annuity of [illegible] for n years, at simple interest at rate i is approximately equal to

$$\frac{1}{i}\log_{\epsilon}\left\{\frac{1+(n+\frac{1}{2})i}{1+\frac{1}{2}i}\right\}.$$

GRADUATED EXERCISES AND EXAMPLES.

PART II.—LIFE CONTINGENCIES (INCLUDING LIFE ANNUITIES AND ASSURANCES).

CHAPTER I.—THE MORTALITY TABLE.

(1).—Explain what is meant by a Table of Mortality, and describe its usual and convenient form.

(2).—If the experience of a given mortality table indicates that, out of 2,000 persons alive at age 30, 29 die before attaining age 31, is it theoretically correct to say that the probability of a person aged 30 dying before age 31 $= \frac{29}{2,000}$?

(3).—Having given a complete table of p_x, accurately representing the probabilities of life at all ages, show how, from the deaths taking place in one year, to calculate approximately the total number living in a stationary population, where there is no disturbance from immigration or emigration.

(4).—If $m_x = \frac{d_x}{\frac{1}{2}(l_x + l_{x+1})}$, show that

$$p_x = 1 - m_x + \tfrac{1}{2}(m_x)^2 - \tfrac{1}{4}(m_x)^3 + \ldots.$$

(5).—(a) Explain the method of forming a table of mortality from the death registers of a place, correcting for the increase of population.

(β) If such correction is disregarded, what would be the effect upon the resulting Mortality Table?

CHAPTER II.—PROBABILITIES OF LIFE.

(6).—If ${}_np_x$ denote the probability that a person aged x will live n years, and p_{x+r} the probability that a person aged $(x+r)$ will live one year, prove that

$$ {}_np_x = p_x \times p_{x+1} \times p_{x+2} \cdot \ldots \times p_{x+n-1}. $$

(7).—Show that $l_{x+n} = {}_np_x(d_x + d_{x+1} + \ldots + d_{\omega-1})$.

(8).—Supposing a given number of marriages contracted between males aged 30 and females aged 25, find the proportion per-cent of the original number who will survive as married couples, widowers, or widows, at the end of 10 years, assuming the probability of dying within 10 years at the age of 30 to be $\frac{265}{2501}$, and at the age of 25 to be $\frac{237}{2611}$.

(9).—Find the following probabilities, namely, that of two lives (x) and (y)

(α) both will not survive n years;
(β) either or both will survive n years.

(10).—Determine (α) the probability that two persons now aged x and y will both die in the nth year from the present time; (β) the probability that one only of them will die in that year.

(11).—Find the value of ${}_{|n}q_{\overline{xy}}$ and ${}_{n-1|}q_{\overline{xy}}$.

(12).—Show that the probability that (x) will survive (n) years, and that (y) will survive $(n-1)$ years $({}_np_x \times {}_{n-1}p_y)$ may be expressed in either of the forms ${}_{n-1}p_{x+1:y} \times p_x$ or $\frac{{}_np_{x:y-1}}{p_{y-1}}$.

(13).—Find the probability that one at least of three lives (x), (y) (z), will fail between the nth and $(n+m)$th year.

(14).—Set down the probabilities of the various contingencies as to death and survival which may happen to three lives during a year (disregarding order of survivorship); and prove the truth of your answer.

(15).—Obtain the probability that out of three lives (x), (y), (z),

(α) one at least will fail in the nth year;
(β) not more than two will fail in the nth year;
(γ) the three lives will fail in the following order: one before the nth year, one in the nth year, one after the nth year.

(16).—Find the probability that at least two lives out of four lives will survive a year. Give a general expression for the probability that at least r lives out of m lives will survive n years.

(17).—If the chance of any one of n persons dying in the course of a year be represented by p, prove that the chance of exactly r of them dying in the course of a year will be $\frac{|\underline{n}}{|\underline{r}\;|\underline{n-r}} \cdot p^r(1-p)^{n-r}$.

(18).—On the supposition in the previous question, prove that the most probable number of deaths in the year is the greatest integer contained in $(n+1)p$.

(19).—What is the probability that, out of seven individuals of a given age, four at least will die in a given time?

(20).—A number n of persons, all of the same age, are each insured for the same sum, £1; the probability of any one of them dying in a year being q, show how to find the probability of the insurance company sustaining a given loss in the course of the year, and prove that the sum of the products of each possible loss into its probability is nq.

(21).—Two offices have each £1,000,000 assured: Office (α) by 100 policies of £10,000 each; Office (β) by 1,000 policies of £1,000 each. Assuming all the ages equal, and the rate of mortality to be 2 per-cent per annum, give an expression for the probability in each case that the claims in one year will amount to £30,000 at least.

(22).—If p be the probability that a person aged x will live one year, find expressions for the following probabilities:—That out of 1,000 persons aged x

(α) exactly 20 will die in a year;
(β) not more than 20 will die in a year;
(γ) 20 designated individuals, and no more, will die in a year;
(δ) 20 designated individuals, at least, will die in a year.

(23).—Six persons, A, B, C, D, E, F, are of the same age, the rate of mortality at that age being 1 per-cent. What is the probability (to five decimal places)

(α) that they will die in an assigned order?
(β) that A will die first, and F last, the remaining order not being fixed?
(γ) that A will die in the first year, and be the first to die?

(24).—Define "rate of mortality" (q_x), "force of mortality" (μ_x), "central death-rate" (m_x), and give expressions showing approximately the relation between these three functions.

(25).—Assuming the force of mortality at age x to be approximately equal to $\frac{d_{x-1}+d_x}{2l_x}$, show that when the decrements of the mortality table are increasing, $\mu_x < q_x$, and when decreasing, $q_x < \mu_x$.

(26).—Show that $\mu_x = -\frac{1}{l_x}\cdot\frac{d}{dx}l_x$.

(27).—Prove that the two expressions for the force of mortality, $-\frac{1}{l_x}\cdot\frac{d}{dx}\cdot l_x$ and $-\frac{d}{dx}\log_\epsilon l_x$, are identical.

(28).—If the force of mortality at age $(x+\frac{1}{2}) = \frac{d_x}{\frac{1}{2}(l_x+l_{x+1})}$, what is the rate of mortality at age x in terms of it?

(29).—Show that $q_x = \frac{1}{l_x}\cdot\int_0^1 l_{x+t}\cdot\mu_{x+t}\cdot dt$.

(30).—Show that $\mu_{xy} = \mu_x + \mu_y$.

Chapter III.—Expectations of Life.

(31).—Define (α) average duration of life, (β) probable lifetime, (γ) mean age at death; and give the value of each in symbols.

(32).—Given the following mortality table, deduce the average duration of life at each age, the average age at death, and the "probable lifetime":—

Age.	No. Living.	Age.	No. Living.
80	116	90	21
81	97	91	15
82	78	92	12
83	64	93	9
84	56	94	7
85	51	95	5
86	39	96	3
87	33	97	1
88	29	98	0
89	24		

(33).—The rate of mortality at each age being given, show how to construct therefrom a table of the mean duration or expectation of life.

(34).—Given a table of mortality, show how to find (α) at what age it is most probable a person of a given age will die; (β) how many years he has an even chance of living.

(35).—Show, upon De Moivre's hypothesis, (α) that the average duration of life at any age equals the probable after-lifetime; (β) that at any age the probability of dying in each of the subsequent years is the same; (γ) that the force of mortality is equal to the rate of mortality at all ages.

(36).—Prove that

$$(1+e_x)=q_x+p_x(1+q_{x+1})+{}_2p_x(1+q_{x+2})+\dots$$

(37).—Find the curtate expectation of a life (x) after n years, and during the following m years $({}_{n|m}e_x)$. What correction would you apply to obtain the expression for the *complete* expectation during the same term?

(38).—Deduce p_x in terms of $\mathring{e}_x$. Prove that if $\mathring{e}_{x-1}=\mathring{e}_x=\mathring{e}_{x+1}$, then $p_{x-1}=p_x$; but that if $p_{x-1}=p_x=p_{x+1}$, then $\mathring{e}_{x-1}$ will not be equal to $\mathring{e}_x$ unless $\mathring{e}_{x+2}=\frac{1}{2}\cdot\frac{1+p_x}{1-p_x}$.

(39).—Show that the average duration of life

$$=\tfrac{1}{2}(q_x+3\,{}_{1|}q_x+5\,{}_{2|}q_x+\dots\dots)$$

(40).—Find an expression for the average age at death of l_x persons.

(41).—Explain what is meant by the curtate expectation of two joint lives (x) and (y). Prove that it is equal to

$$p_{xy}+{}_2p_{xy}+{}_3p_{xy}+\dots\dots$$

Chapter IV.—Probabilities of Survivorship.

(42).—Find the probability that (x) will die in the nth year, (y) being alive at the moment of (x)'s death.

(43).—Show that the probability of (x) dying before (y) in the nth year, when no assumption is made as to the distribution of deaths throughout the year, may be expressed in the form

$$\frac{d_{x+n-1}l_{y+n-1}}{2l_xl_y}+\frac{d_{y+n}d_{x+n-1}-d_{y+n-1}d_{x+n}}{12l_xl_y}.$$

(44).—Find the probability that (x) will die before (y).

(45).—Assuming that the deaths in each year are uniformly distributed, and supposing that two persons (x) and (y) both die in the same year, prove that the chance of (x) dying before (y) is accurately $\frac{1}{2}$.

(46).—Show that the probability of a life (x) dying before a life (y) may be expressed by the formula

$$\frac{l_{xy}+N'_{x-1:y}-N'_{x:y-1}}{2l_{xy}}.$$

(47).—Find the probability of a person aged x dying before one aged y, or within t years after.

(48).—Find expressions for the following probabilities:—

(α) That (y) will be alive t years after the death of (x).

(β) That (y) will be alive at the end of the tth year succeeding the year of the death of (x).

(49).—Find the value of Q^1_{xyz}, Q^2_{xyz}, and Q^3_{xyz}.

CHAPTER V.—STATISTICAL APPLICATIONS OF THE MORTALITY TABLE.

(50).—Given the values of l_x and e_x at every age, show how to construct the columns L_x, N'_x, T_x, and Y_x, and explain generally the uses to which such columns may be applied.

(51).—A community proposes to establish pensions for such of their number as attain a certain age, to be provided by equal annual subscriptions payable up to that age. How would you proceed to find the amount of such subscriptions?

(52).—A body of men, kept of uniform strength by annual entrants engaged at age 20, and superannuated at age 60, are grouped in three classes, the members in the classes being as three in the junior, two in the intermediate, and one in the senior. Assuming promotion to go by seniority, how can a mortality table be used for finding the average age of promotion from one class to another? What causes may be in operation to vitiate the results?

(53).—Show that in a stationary population the average present age at any time is equal to the average expectation of life at age 0 and upwards.

(54).—In a stationary community there are 209 deaths annually to each 10,000 inhabitants. What is the average age at death?

CHAPTER VI.—FORMULAS OF DE MOIVRE, GOMPERTZ, AND MAKEHAM, FOR THE LAW OF MORTALITY.

(55).—Explain any advantages which would accrue from being able to express the number living at any age, out of a given number born, as a function of the age, and give examples of any formulas that have been suggested for this purpose.

(56).—If a mortality table were such that the numbers living formed a series in geometrical progression, what would be the nature of the tables giving the chance of dying in a year, and the expectation at any age?

(57).—Upon what hypothesis as to the law of mortality is Gompertz's formula based? Prove that if this hypothesis is assumed, the formula of Gompertz for the number living at a given age x will be obtained.

(58).—Upon what modification of Gompertz's hypothesis as to the law of mortality is Makeham's formula based? Show that by adopting this modified hypothesis, the formula of Makeham for the value of l_x will be obtained.

(59).—Given $l_x=ks^x(g)^{c^x}$, find the values of $\log p_x$ and of μ_x.

(60).—Given a table of the values of $\log l_x$ at all ages, state generally how you would proceed to deduce values for the constants k, s, g, and c, involved in Makeham's formula, $l_x=ks^x(g)^{c^x}$.

CHAPTER VII.—ANNUITIES AND ASSURANCES.

(61).—Find by the Life Table given in the "Text-Book", taking interest at 3 per-cent, the present value of a sum to be received by a person aged 10 provided he be living at age 21.

(62).—Find the value of an annuity on a single life, and show how the annuity at any age can be expressed in terms of that at the next higher age.

(63).—Prove that the value of £1, to be received immediately upon the decease of a person of a given age, is greater than that of £1 receivable at the expiration of a number of years certain equal to the complete expectation of life at that age.

(64).—Assuming that the present value of a life annuity has been demonstrated, show clearly and logically what is the present value of 1, payable at the end of the year of death, finding equations connecting a_x and A_x in terms of v, d, and a_∞ respectively.

(65).—Given the formula $1 = ia_x + (1+i)A_x$, find an analogous expression involving the values of temporary benefits, and thence deduce the value of a term assurance.

(66).—Give a verbal interpretation of the formulas

$$(\alpha)\quad {}_{|n}A_x = v(1 + {}_{|n-1}a_x) - {}_{|n}a_x$$

$$(\beta)\quad A_{x\overline{n}|} = v - d\,{}_{|n-1}a_x.$$

(67).—Prove that the single premium for an assurance payable on a life now aged x years attaining the age of $(x+t)$ or dying previously may be expressed by the formula $\dfrac{1 - i\,{}_{|t-1}a_x}{1+i}$, and show that this is equal to $\dfrac{M_x - M_{x+t} + D_{x+t}}{D_x}$.

(68).—Describe the construction of the D and N columns for joint lives proposed by Professor de Morgan, and applied by Dr. Farr to the English Life Table No. 3, and point out in what respects it differs from that previously used.

(69).—Give formulas for the following, using commutation symbols:—

- (α) Annual premium payable during m years only for an endowment assurance on (x) payable at $(x+m+n)$ or previous death.
- (β) Single premium for a temporary assurance, payable in the event of (x) dying within n years.
- (γ) Annual premium for a whole-life assurance on (x), deferred n years, the premium payable (i) for n years, (ii) for the whole of life.

(70).—If there be n lives all of the same age x, find the value of an annuity on the last survivor.

(71).—Find, in terms of joint-life annuities, an expression for the value of an annuity payable only while *exactly* r lives out of m lives survive.

(72).—Obtain the value of an annuity, payable so long as two at least out of three lives (x), (y), (z) are alive. Obtain this value also by general reasoning.

(73).—Find a formula to express the amount to which a life-annuity-due will, on the average, accumulate at the end of the year of death, supposing each payment to be invested and accumulated at compound interest. Explain the difference between this value and $\frac{N_{x-1}}{M_x}$.

(74).—Find the value of an annuity on (x) during the lifetime of (y), and for t years after the death of (y), if (x) live so long.

(75).—Given a table of annual premiums, show how to construct a corresponding table of mortality.

(76).—Obtain the values of a_x and A_x upon De Moivre's hypothesis as to the rate of mortality.

(77).—Give an algebraical proof of the formula

$$a_{\overline{abc}:\overline{xyz}} = a_{\overline{abc}} + a_{\overline{xyz}} - a_{\overline{abcxyz}}$$

and show by argument that the result is correct.

(78).—Give formulas for the value of an endowment, payable

(α) If (x) survive n years.
(β) If both (x) and (y) survive n years.
(γ) If either (x) or (y) survive n years.
(δ) If (x) survive, and (y) die within n years.

(79).—Investigate formulas for annuities payable until the death of the last survivor of (x) and (y), the payments being reduced to $\frac{m}{n}$ of their former amount (α) at first death, (β) in the event only of (x) dying before (y).

(80).—Prove that $A_{\overline{xxx}} = A_{xxx} - 3(A_{xx} - A_x)$. Does the annual premium follow the same rule? Prove your answer.

(81).—Find the value of a temporary annuity payable for m years if (x) survive, or for n years if (y) survive, (x) having died within the n years. Assume $n < m$.

(82).—The yearly rents of an estate belong in equal shares to four sisters, being equally divided at the end of each year amongst all of the sisters who survive that year, the last survivor taking the whole for the rest of her life. Find in terms of the single and joint-life annuities an expression for the present value of the interest of any one of them. Write down the general formula for the same thing, when the number of persons is n.

CHAPTER VIII.—CONVERSION TABLES FOR SINGLE AND ANNUAL ASSURANCE PREMIUMS.

(83).—If P represents the present value of £1 to be received at the end of the year in which a certain status may fail, and Q the present value of £1 per annum to be received during its continuance, prove that $P=\frac{1-iQ}{1+i}$. Hence show that $(1+i)M_x+iN_x=D_x$.

(84).—Give a general formula for the class of benefits the single and annual premiums for which can be derived from Conversion Tables, and explain it verbally.

(85).—Show how Conversion Tables can be employed in finding the single and annual premiums for an endowment assurance.

(86).—Show how to construct a table from which may be found the annual premium corresponding to any given single premium on a whole life assurance.

(87).—Given the values of the single and annual premiums for the insurance of £1 on a life (x), find the rate of interest.

(88).—By the H^M Table, the value of an annuity on a life aged 30 is 17·1309, and the single premium for an assurance on the life is ·302658. Find, without referring to any book, the rate of interest at which these values are calculated.

(89).—The mortality by a given table is such that the values of annuities at all ages, at the rate of interest i per-cent, are identical with those of another table at the rate of i' per-cent. Will the corresponding tables of annual premiums also coincide? Give a reason for your answer.

CHAPTER IX.—ANNUITIES AND PREMIUMS PAYABLE FRACTIONALLY THROUGHOUT THE YEAR.

(90).—Assuming a uniform distribution of deaths throughout the year, prove the formula for the value of an annuity payable half-yearly

$$\frac{a_x}{4}\left(2+\sqrt{v}+\frac{1}{\sqrt{v}}\right)+\frac{\sqrt{v}}{4},$$

and show that the common approximation $(a_x+\frac{1}{4})$ agrees with the above to two decimal places, but not in the third place.

(91).—The common rule for finding the value of an annuity payable half-yearly or quarterly is to add $\frac{1}{4}$ or $\frac{3}{8}$ to the value when payable yearly. What is the amount of the error?

(92).—Show that the value of a life annuity payable m times a year, the first payment being made after the interval $\frac{1}{2m}$, is approximately equal to $(a_x+\frac{1}{2})$ whatever value is assigned to m.

(93).—Given the formula for converting an annuity payable yearly into one payable m times a year, investigate a rule for finding a premium payable m times a year from the annual premium.

(94).—If P be the annual premium for the insurance of 1 on a life (x), show that the half-yearly premium is nearly equal to $\frac{\frac{1}{2}P_x}{1-\frac{1}{4}(P_x+d)}$.

(95).—State a general formula for the value of an annuity on a single life or combination of lives, payable m times a-year, and apply it to determine the value of a continuous reversionary annuity to (x) after (y).

(96).—Find the annual premium for a half-yearly annuity on (x) deferred n years, the first payment of the annuity being made six months after the payment of the last premium.

(97).—Assuming the usual correction for the value of an annuity payable quarterly, prove that the value of a temporary annuity on a life (x) payable quarterly for n years is $\frac{N_x-N_{x+n}-\frac{3}{8}D_{x+n}}{D_x}+\frac{3}{8}$.

CHAPTER X.—ASSURANCES PAYABLE AT ANY OTHER MOMENT THAN THE END OF THE YEAR OF DEATH.

(98).—Find the single premium for an assurance on a life (x) payable at the instant of death.

(99).—What will the formula $A_x=\frac{1-ia_x}{1+i}$ become when the annuity is payable (1) half-yearly, (2) m times a year, (3) continuously?

(100).—Prove that, upon the assumption of a uniform distribution of deaths throughout the year, the value of an insurance payable at the instant of death is equal to $\frac{i\mathrm{A}_x}{\log_\epsilon(1+i)}$. Expand this expression in a series as far as terms containing i^3.

(101).—Find the value of $\bar{\mathrm{A}}_{x\overline{n}|}$.

(102).—Obtain an expression for the value of the force of mortality in terms of the expectations of life.

CHAPTER XI.—COMPLETE ANNUITIES.

(103).—Find an expression for the value of a continuous annuity on a life aged x, and thence deduce a formula for the value of $\mathring{a}_x$.

(104).—What is the nature of the error involved in the approximate formula for the value of a complete annuity payable yearly—

$$\mathring{a}_x=a_x+\tfrac{1}{2}\mathrm{A}_x(1+i)^{\frac{1}{2}}?$$

Deduce a more correct expression, on the assumption of a uniform distribution of deaths, and thence obtain an approximate value for the amount of the error involved in the above formula.

(105).—What addition should be made to the value of a curtate annuity payable yearly (a_x) to obtain the value of a complete annuity payable half-yearly ($\mathring{a}_x^{(2)}$)?

(106).—Given the formula for a complete annuity payable m times a year

$$\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}\bar{\mathrm{A}}_x-\frac{\mu_x}{12m^2},$$

obtain an expression for the value of $({}_{t|}\mathring{a}_{xyz}^{(4)})$, a complete annuity deferred t years, and payable quarterly during the joint existence of (x), (y), and (z).

(107).—Show that approximately

$$\mathring{a}_x\doteq a_x+\tfrac{1}{2}\bar{\mathrm{A}}_x-\tfrac{1}{12}\bar{\mathrm{A}}^{1}_{x\overline{1}|}.$$

CHAPTER XII.—JOINT-LIFE ANNUITIES.

(108).—State some of the methods which have been suggested for the approximate calculation of the value of an annuity on three joint lives. Which would you select as likely to give the most satisfactory result without excessive labour?

(109).—Show that if $a_{yz}=a_{w}$, the value of an annuity on the last survivor of three lives, (x), (y), (z), is approximately equal to $a_{\overline{xy}}+a_{\overline{xz}}-a_{\overline{xw}}$.

(110).—Show that, if Gompertz's formula $l_x=k(g)^{c^x}$ be assumed, the probability of two joint lives (x) and (y) surviving n years is identical with the probability of a single life (w) surviving the same period, where $c^w=c^x+c^y$.

(111).—Show that, if ${}_tp_x=s^t(g)^{c^x(c^t-1)}$, where s, g, and c are constants independent of x and t, the probabilities of life of n joint lives x, y, z, ..., are identical, for all values of t, with the probabilities of life of n joint lives of a uniform age w, where $nc^w=c^x+c^y+c^z+\ldots\ldots$

(112).—Prove that, upon Makeham's hypothesis as to the law of mortality $l_x=ks^x(g)^{c^x}$, the value of an annuity on m joint lives x, y, z, ..., calculated at the rate of interest i, is equal to the value of an annuity on m joint lives of a uniform age w, calculated at the rate of interest i', where $i'=\frac{1+i}{s^{m-1}}-1$.

(113).—How may the relation shown in the previous example be practically applied in the approximate calculation of the value of an annuity on three lives at a rate of interest of 3 per-cent, by means of tables of the values of annuities on two joint lives, calculated at 3, 3½, and 4 per-cent?

CHAPTER XIII.—CONTINGENT, OR SURVIVORSHIP, ASSURANCES.

(114).—Prove the formula for a contingent assurance

$$A^{1}_{xy}=\frac{1}{2}\left(A_{xy}+\frac{a_{x-1:y}}{p_{y-1}}-\frac{a_{x:y-1}}{p_{y-1}}\right).$$

What supposition is made, in obtaining this formula, as to the deaths which take place in the course of any year?

(115).—Find the value of an assurance on the life of (x) provided he die after (y). For which of the two lives would the usual medical examination be required?

(116).—Find the single premium for an assurance for n years of 1 on the failure of a life aged x, provided that another aged y survive him.

(117).—Write down, in commutation symbols, the single premium for a survivorship assurance, using Professor de Morgan's method of constructing the joint commutation columns, and also the ordinary method previously in use.

(118).—Having given complete D and N columns for single and joint lives, and also a table of M^1_{xy} for $x<y$; find $|_tA^1_{xy}$, $|_tA_{x\overset{1}{y}}$, $|_tA^2_{xy}$ and $|_tA_{x\overset{2}{y}}$.

(119).—Find the single and annual premiums for an assurance payable in the event of (x) dying in the lifetime of (y), or within t years from the death of (y).

(120).—How would you approximate to the annual premium for an assurance payable on the death of a person aged 48, in the event of his dying before the survivor of two persons aged respectively 75 and 70, or within one year after the death of such survivor?

(121).—Find a formula for the annual premium to assure a payment on (x) attaining the age of $(x+n)$ or at previous death, provided (y) be living in either case; and show how it can be adapted for the use of the values tabulated in the Institute Life Tables.

(122).—Show how to find the single premium for an assurance payable on the death of B aged y, provided he die after A aged x, and before C aged z. If the premium is to be paid annually until the risk determines, by what annuity would you divide the single premium?

(123).—Determine $A^{1,3}_{x:yz}$, the present value of £1 payable upon (x) failing either the first or last of three lives (x), (y), and (z).

(124).—Show that $\overline{A}^1_{xy} = \mu_x\bar{a}_{xy} - \dfrac{d\bar{a}_{xy}}{dx}$, and hence deduce the approximate expression for the value of the annual premium corresponding to this single premium $\overline{A}^1_{xy}$, namely, $\mu_x + \frac{1}{2}\dfrac{a_{x-1:y} - a_{x+1:y} - \mu_x}{a_{xy}+1}$.

CHAPTER XIV.—REVERSIONARY ANNUITIES.

(125).—Show that $a_y - a_{xy} = \dfrac{d_xN_y + d_{x+1}N_{y+1} + \cdots}{l_xD_y}$

(126).—Find the value of an annuity for such portion of a term of t years certain from the present time as will remain after the death of a person aged x, the first payment to be made at the end of the year of his death.

(127).—Give an expression for the annual premium for a contingent annuity to commence at the death of A, aged x, and to continue as long as either B or C, aged respectively y and z, is living.

(128).—An annuity on the longest of three lives, A, B, C, aged respectively x, y, and z, is to be enjoyed by A during his life; after his decease it is to be divided equally between B and C during their joint lives, and the survivor of them is to have the whole. Find an expression for the value of B's interest.

(129).—A sum S is applied in the purchase of an annuity on three lives aged (x), (y), (z); the annual payment being A while all three are alive, $\frac{4}{5}$A while two are alive, and $\frac{3}{5}$A while one is alive. Find A.

(130).—Required the single and annual premiums for an annuity payable to the last survivor of (x) and (y), to commence at first death if within n years; or at the end of n years, if both be then living.

(131).—Find single and annual premiums for an annuity of 1 during the term which may remain between the period of y's death and that of x completing the age $(x+n)$.

(132).—Find the value of an annuity on (x), to commence at the death of (y) and to continue payable till the end of t years from now, and as much longer as (x) may live.

(133).—Find the value of an annuity on a life aged y, the first payment to be made at the end of the year of death of a person aged x, but the annuity to continue for t years certain whether (y) survive or not.

(134).—Find in a convenient form for computation the single and annual premiums for an annuity to commence on the death of (y) and continue payable during the remainder of the life of (x), but to be payable only if (y) dies within t years.

(135).—There are two expressions for the value of a reversionary annuity to (x) after (y), namely:—

$$\Sigma v^n({}_np_x - {}_np_{xy}) \text{ and } \Sigma v^n \frac{l_{x+n} d_{y+n-1}}{l_x}(1+a_{x+n})$$

Give an algebraical demonstration that these are identical.

(136).—How would you proceed to obtain the value of a reversionary annuity to (a) after (b) in the case where the two lives are now resident in India, but (a) has the intention of living in England after the death of (b)?

(137).—(α) Prove that $(a_x - a_{xy}) : (A_y - A_{\overline{xy}}) :: (1+i) : i$. (β) Explain clearly in what respects the assumptions involved in the ordinary formula for a reversionary annuity to (x) after the death of (y), namely, $(a_x - a_{xy})$, do not agree with the conditions of practice, and give particulars of the various formulas devised to remedy this.

(138).—The formula $(a_x - a_{xy} - \frac{1}{4}A^{1}_{xy})$ has been given for the value of a half-yearly reversionary annuity payable during the life of (x) after the death of (y). On what assumption is this solution approximately correct?

(139).—Deduce a formula for the value of a complete annuity to (x), payable m times a year, to be entered upon at the moment of the death of (y), $=\mathring{a}^{(m)}_{y|x}$.

CHAPTER XV.—COMPOUND SURVIVORSHIP ANNUITIES AND ASSURANCES.

(140).—Find the present value of an annuity on a life (x) after the failure of the joint existence of (y) and (z) provided that event take place by the death of (z).

(141).—Show that, upon Gompertz's hypothesis as to the law of mortality, the relation ${}_{|t}q^{1}_{yz} = Q^{1}_{yz} \cdot {}_{|t}q_{yz}$ is accurately true.

(142).—Find the value of a contingent life interest for the lives of (x) and (y) and the survivor, after the death of (z), subject to the condition that (x) shall survive (z).

(143).—Investigate a formula for the single premium for an assurance payable on the death of (x) if he die third of the three lives, (x), (y), and (z), (z) having died first. $(A^{3}_{xy\underset{1}{z}})$.

CHAPTER XVI.—COMMUTATION COLUMNS, AND THEIR APPLICATION TO VARYING BENEFITS, AND TO RETURNS OF PREMIUM.

(144).—Explain a commutation table, and the relations of the columns to each other. Why is it called a commutation table? In what way does the N of the English Life Table differ from Davies's $\dot{N}$?

(145).—Given only the D and N columns, show how to deduce all the n them.

—The value of a deferred annuity is $\frac{N_{x+t}}{D_x}$. Explain fully the of the expression $\frac{N_{x-t}}{D_x}$:

(147).—Having given a_x, and the value of the annuity on (x) commencing at k payable at the end of the first year, and increasing h per annum; find the value of the corresponding assurance commencing at k if death occur in the first year, and increasing h per annum.

(148).—Find the present value of an annuity on (x) payable half-yearly, commencing at £1, the payments to be doubled every 10 years.

(149).—Investigate a formula for the annual premium for an assurance on the life of (x) which is to be 1, 2, 3, . . . n pounds, according as the death of (x) shall take place in the 1, 2, 3, . . . nth year, and to remain constant at the last-named amount during the remainder of life, (α) for a premium payable n times if (x) live so long, (β) for a premium payable throughout life.

(150).—State a formula for the net annual premium for an assurance on a life (x) commencing with 1 and decreasing ·05 every year until its extinction at the end of 20 years; the premium being also reduced by $\frac{1}{20}$th of its original amount each year.

(151).—Explain the method of calculating a table of ascending premiums, giving the proper formulas. What precaution must always be observed in practice with regard to the magnitude of these premiums, and how would you fix the premium to be charged during the first term of years?

(152).—Deduce a general formula (in terms of the commutation columns) for the value of an annuity on a life (x), whose successive payments are u_0, u_1, u_2, &c. . . . . .

(153).—Find the annual premium for a deferred annuity on (x) to commence n years hence, it being stipulated that if (x) die before the annuity commences, the net premiums paid are to be returned with compound interest.

(154).—Show in the previous question that if the rate of interest at which the premiums are calculated is the same as that at which they are accumulated, say i, then the net annual premium $= \dfrac{via_{x+n}}{(1+i)^n-1}$.

(155).—Prove that the single premium for an assurance of 1 on the life of x, with the return of the premium along with the sum assured, is equal to the annual premium to secure a perpetuity of 1 per annum, of which the first payment is made at the end of the year in which x shall die.

(156).—Find the net annual premium for an assurance of 1 payable at the end of $(m+n)$ years if (x) be then living, or at his death if that happen after m years but before $(m+n)$ years: it being stipulated that if (x) die within m years all the office premiums are to be returned except the first.

(157).—Find the net annual premium for an assurance on (x) of 1, with return of all the office premiums paid, the premiums being payable for t years only.

(158).—Find the annual premium payable for n years for an assurance on the life of (x), with return of the excess of the limited premium over the ordinary whole of life rate (pure premiums only) in the event of death occurring in the first n years.

(159).—A survivorship annuity of 100 payable half-yearly till death on (26) after (32) is to be paid for by an annual premium, returnable if (26) die before (32). Required the annual premium which shall give to the grantor a profit of 15 per-cent, the rates of mortality and interest being assumed as Carlisle 4 per-cent.

(160).—Find the single premium for an annuity to (x) after (y), with the condition that the premium be returned if (x) die before (y).

CHAPTER XVII.—SUCCESSIVE LIVES.

(161).—Show that the value of 1 to be received on the failure of the successive lives x and y is $A_x \times A_y$. When is the second life supposed to be nominated?

(162).—An estate is held for a single life, renewable for single lives successively, by payment of a fine of £1 at the end of the year in which the life in nomination fails: assuming that the present life is now aged x, and that the succeeding lives are all nominated at the age y, find the present value of all the fines to perpetuity.

(163).—A copyhold estate is held on three lives aged severally 61, 50, and 45, each renewable at the end of the year in which it drops by a life of 7 years of age on payment of a fine of £5. Give an expression for the present value of all the fines for ever.

—A perpetual annuity is to be enjoyed, first by a person aged x
fe, afterwards by a successor to be appointed at his death, and
second life fails, by a third to be then appointed, and so on.
the present value of the annuity for the first n successive
ages being all different?

(165).—Investigate an expression for the value of the nth presentation to a living.

(166).—Find the value of the second, and every third succeeding presentation to a living, assuming all the nominees to be of the same age on presentation.

Chapter XVIII.—Policy-Values.

(167.)—Show that the values of policies which have been n years in force are equal to the accumulated premiums less the accumulated claims.

(168).—If l_x persons each buy an endowment at an annual premium, show that the total amount which will be paid to the survivors is made up of an accumulation of (α) the premiums paid by the survivors; (β) the premiums paid by those that die.

(169).—Find the value of a policy in terms of (α) annual premiums and rate of interest; (β) annuity and annual premiums; (γ) single premiums at entry and at valuation.

(170).—Explain under what conditions ${}_nV_x < {}_{n-1}V_{x+1}$ and give an example from some known table of mortality where the anomaly occurs.

(171).—Give a verbal interpretation of the expression

$$(P_x+d)(1+a_{x+n})+{}_nV_x=1.$$

State what this becomes when the premiums are payable for a limited number of years.

(172).—If P_x denote the annual premium for a whole-life insurance, $P_{\overset{1}{x}\overline{n}|}$ the annual premium for a term insurance for n years, and $P_{x\overset{1}{\overline{n}|}}$ the annual premium for an endowment payable at the end of n years, prove that the value of a policy effected at the age x which has been n years in force may be expressed by the formula ${}_nV_x = \dfrac{P_x - P_{\overset{1}{x}\overline{n}|}}{P_{x\overset{1}{\overline{n}|}}}$.

(173).—If a whole-life policy be valued by the "hypothetical" or "re-assurance" method, show under what circumstances the reserve will be (α) greater than, (β) less than, (γ) equal to the policy-value by the ordinary net-premium method.

(174).—Given two mortality tables, how would you proceed to ascertain within what limits as to entry-age and duration the policy-values brought out by the one table are greater or less than those brought out by the other table?

(175).—Prove that, if the annuity-values by two several tables of mortality are in the relation $a'_x=(1+\kappa)a_x$, where κ is a constant, then the policy-values by the two tables will be equal for all ages and durations. What will be the relation between the probabilities of living a year at any age x (p_x) by the two tables?

(176).—In two mortality tables, (A) and (B), the probability of living a year by the latter table is throughout less in a constant ratio than by the former. What relation will exist between the policy-values by the two tables?

(177).—If the rate of mortality in one table be throughout greater than that in another table, should you expect the values of policies obtained in the ordinary way from the first table to be greater or less than those obtained from the second? State your reasons.

(178).—Given a table of policy-values $({}_nV_x)$ for all integral values of x and n, how would you proceed to ascertain the value of a whole-life policy, effected by yearly premiums at age x, after a duration of $\left(n+\frac{t}{m}\right)$ years? $\left({}_{n+\frac{t}{m}}V_x\right)$.

(179).—What modification should be introduced in the usual formula $\Sigma A_{x+n}-\Sigma P_x(\frac{1}{2}+a_{x+n})$ for valuing a group of policies effected n years ago, with yearly premiums at age x, in the case where, from an unequal distribution of premium income, it is ascertained that the yearly payments next falling due will, on the average, be payable $7\frac{1}{2}$ months hence?

(180).—Show to what extent the payment of claims by an assurance company immediately on proof, instead of at the end of the year of death, affects (α) the liability under the sum assured, (β) the asset in respect of the annual premiums: and state what additional reserve is necessary in each case.

(181).—Prove that the value, after n years, of an endowment-assurance policy effected on a life (x), payable at $(x+n+m)$ or previous death, may be expressed by either of the following formulas:—

$$\frac{P_{x+n:\overline{m}|}-P_{x:\overline{n+m}|}}{P_{x+n:\overline{m}|}+d}\,;\qquad\qquad \frac{A_{x+n:\overline{m}|}-A_{x:\overline{n+m}|}}{1-A_{x:\overline{n+m}|}}.$$

—Apply the *retrospective* method to the valuation of an endow-
icy effected n years ago at age x, payable at age $(x+n+m)$,

with the condition that the whole of the office premiums are to be returned in the event of the life failing during the term of the endowment, and prove the identity of the expression thus obtained with that which would be arrived at by valuing separately the future benefit assured, and the future net-premium payments.

(183).—Show that the value of an endowment policy taken out at age x, payable in n years, is, after it has been t years in force,

$$\frac{P_{\overset{1}{x}\overline{n}|}}{P_{\overset{1}{x}\overline{t}|}}.$$

(184).—Show that the amount of a free or "paid-up" policy which may be given in exchange for a whole-life assurance effected on a life (x), after n years' duration, is equal to $\left(1-\frac{P_x}{P_{x+n}}\right)$, and give a verbal interpretation of this expression.

(185).—In a society consisting of n members, 1 is payable at the death of each member by the survivors. Find an approximate expression for the present value of the future payments.

(186).—How would you proceed to value as for the 19 May 1877 a policy dated 19 August 1848 for £100 on a life then aged 41, at a premium of £2. 5*s*. 2*d*. for the first seven years, £3. 9*s*. 9*d*. for the second seven years, and £4. 14*s*. 4*d*. for the remainder of life?

(187).—The values of all the policies in an office at the beginning of any year and the premiums for that year, both accumulated at interest for one year, are equal to the claims of the year and the values of the policies at the end of the year. Under what conditions is this true; and show what modifications would be required on the assumption that the premiums are payable throughout the year?

CHAPTER XIX.—LIFE INTERESTS AND REVERSIONS.

(188).—Investigate a formula for the market value of a life interest in a term annuity.

(189).—Give the formula for the value of a life interest of £s per annum, and state for what amount the life policy shall be effected. What is the value of such policy to the purchaser of the life interest, after the expiration of t years?

(190).—If a'_x represent the value of a life-interest secured by an assurance effected at a given annual premium, show that at the expiration of n years the value of the policy (*i.e.*, the value of the sum assured minus the value of the premiums payable) is $S\left(1-\frac{1+a_{x+n}}{1+a'_x}\right)$ where S denotes the amount of the assurance; and further, that this expression resolves itself into $(a'_x - a_{x+n})$.

(191).—A common formula for the value of a reversionary life-interest is

$$\frac{1-(P_x+d)(1+a_{xy})}{P_x+d}$$

If £1 is advanced in consideration of a reversionary charge so calculated, set out the theoretical sum assured, the theoretical cost of the annuity purchased, and the value of the charge at death of (y) coupled with the policy.

(192).—Find a formula for the value of a reversionary life-interest so as to return one rate of interest while the life-interest is in reversion, and another rate when it is in possession.

(193).—A, aged x, is entitled to a reversionary life interest contingent on his surviving (y) and (z), and neither leaving issue. Give the formula for valuing A's interest, allowing for a single whole-world premium of m per-cent, and a single issue premium of n per-cent.

(194).—A, aged x, is entitled to a sum S provided he survive B, aged y. Find the amount which a purchaser could give for the reversion.

CHAPTER XX.—SICKNESS BENEFITS.

(195).—In a given community the rate of sickness (average number of weeks' sickness per annum) at each age, z_x, $=A+Bq_x$, where q_x=the rate of mortality at age x. Express the value of the sickness benefit ceasing at age 70.

(196).—If, in two sickness tables, (A) and (B), the rates of sickness hroughout identical, while the probabilities of life at each age table (B) are less than those in table (A) in the ratio)p_x, what relation will hold between the values of the sickness nder the two tables?

--If the law of sickness be such that at any age two are constantly sick for one that dies, show that the single premium for a sick allowance of 10*s*. per week at age x, to cease at age 65, is

$$52{\cdot}18\left(\frac{\overline{M}_x-\overline{M}_{65}}{D_x}\right).$$

(198).—Given a table showing at each age the number living (l_x) and the average number of weeks' sickness (z_x), how would you proceed to deduce the present value of a sick allowance of 1 per week (α) payable throughout life, (β) payable up to age 70, (γ) to commence at age 60 and continue throughout the remainder of life?

(199)—How would the formula for the present value, at age x, of a sick allowance throughout life, be modified, in the case where it is assumed, that a member will be permanently on the sick list after age 70?

(200).—A friendly society grants the following benefits to its members: (α) a sum of £10 at death; (β) a weekly allowance of £1 during the first six months' sickness, 10*s*. during the second six months' sickness, and 5*s*. during subsequent sickness up to age 70; (γ) a permanent allowance of the minimum sickness rate from age 70 until death. State a formula for the weekly contribution necessary to provide the above benefits at age x, and for the reserves which should be in hand after n years, assuming that 30 per-cent of the weekly contribution is absorbed in expenses.

CHAPTER XXI.—CONSTRUCTION OF TABLES.

(201).—Give a practical method of forming a table of the values of

$$\log\tfrac{1}{2}(p_x^{-1}-1)(p_y^{-1}+1).$$

(202).—Give a method of calculating the D and N columns by a continued process.

(203).—What, in your opinion, would be the best way, as regards speed and accuracy, to construct a table of the values of annuities, having given the usual l_x column of a life table? If you are acquainted with more than one method of constructing the annuity table, what do you consider to be the special advantages of each?

(204).—Give De Moivre's method of calculating the values of annuities, and show the applicability of Gauss's logarithmic tables.

(205).—Explain the formula

$$B = v\pi\left(\frac{p}{\pi} + B_1\right)$$

and apply it to the construction of a reversion to £1 upon the decease of a single life.

(206).—What conditions must the benefits fulfil to which the demonstration of the above formula applies, and in regard to which it subsists?

(207).—Given a complete table of values of deferred annuities (${}_{n|}a_x$) and of temporary annuities (${}_{|n}a_x$), for all values of x and n, how would you proceed to verify the results?

(208).—Describe how to construct a table of policy-values of whole-term assurances.

(209).—Prove the formula ${}_nV_x = 1 - (\pi_x + d)(1 + a_{x+n})$, and point out its advantages in calculating a table of policy-values.

(210).—Show how to construct by a continued method tables of policy-values of endowment assurances. What precautions would you take to check the results?

(211).—Show how by a continued process to construct commutation columns M^1_{xy} for survivorship assurances.

(212).—Deduce a formula for the continued construction of a table of single premiums for survivorship assurances on (x) against (y).

(213).—Describe at length a method of calculating a complete table of last-survivor annuities.

(214).—Explain how you would construct tables of the values of $\bar{a}_x$ and $\bar{A}_x$.

(215).—Show how to construct a table of premiums for the assurance of 1 for the term of one year by a continued process.

Chapter XXII.—Formulas of Finite Differences.

(216).—If u_x be any function of x of n dimensions, prove that constant, and hence show how to form a table of cubes of al numbers expeditiously.

—Prove that $\Delta^n x^n = \lfloor n$.

—Find $\Delta^n a^x$, when x is variable, the increment of x being unity.

—By means of the formula

$$\Delta^m 0^n = m\{\Delta^{m-1} 0^{n-1} + \Delta^m 0^{n-1}\},$$

[Solution

or otherwise, form a table of the differences of the powers of nothing, as far as $\Delta^7 0^7$.

(220).—Investigate the expressions

(α) for u_{x+n} in terms of Δu_x, $\Delta^2 u_x$, $\Delta^3 u_x$, &c.

(β) for $\Delta^n u_x$ in terms of u_x and its successive values.

(221).—Show that

$$\Delta^n u_x = u_{x+n} - \frac{n}{1} u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \ldots\ldots$$

Hence show that, second differences being constant,

$$\Delta u_{x+2} - 2\Delta u_{x+1} + \Delta u_x = 0.$$

(222).—If $\Delta x = a$, and $\Delta u_x = u_{x+a} - u_x$, prove that

$$u_{x+n} = u_x + \frac{n}{a}\Delta u_x + \frac{n(n-a)}{2a^2}\Delta^2 u_x + \frac{n(n-a)(n-2a)}{2.3a^3}\Delta^3 u_x + \text{\&c.}$$

(223).—Obtain a formula for expressing u_{x+m} in terms of u_x and the successive finite differences Δu_x, $\Delta^2 u_x$, &c. If $u_1=4$, $u_2=30$, $u_3=120$, $u_4=340$, $u_5=780$, and $\Delta^4 u_x$ is constant, find an algebraical expression for u_x.

(224).—Express $u_{x+\frac{1}{n}}$ in terms of u_x and its successive differences.

(225).—(α) Given $(n+1)$ consecutive equidistant values, u_0, u_h, u_{2h}, ... u_{nh}, of u_x, which is a rational integral algebraic function of x of degree n, find the general expression for u_x.

(β) When u_x is an expression of the 4th degree in x, and $u_0=0$, $\Delta u_0=1$, $\Delta^2 u_0=14$, $\Delta^3 u_0=36$, $\Delta^4 u_0=24$, find u_x, the increment of x corresponding to the Δ differences being h, so that $u_{x+h}-u_x=\Delta u_x$.

(226).—In the series x^5, $(x+h)^5$, $(x+2h)^5$, $(x+3h)^5$, $(x+4h)^5$, ..., let u_x represent the first term, and h be written Δx, and show that $(x+n)^5$, where n is a multiple of h, is equal to

$$u_x + \frac{n}{\underline{|1}}\cdot\frac{\Delta u_x}{\Delta x} + \frac{n(n-h)}{\underline{|2}}\cdot\frac{\Delta^2 u_x}{(\Delta x)^2} + \frac{n(n-h)(n-2h)}{\underline{|3}}\cdot\frac{\Delta^3 u_x}{(\Delta x)^3}$$

$$+ \frac{n(n-h)(n-2h)(n-3h)}{\underline{|4}}\cdot\frac{\Delta^4 u_x}{(\Delta x)^4}$$

$$+ \frac{n(n-h)(n-2h)(n-3h)(n-4h)}{\underline{|5}}\cdot\frac{\Delta^5 u_x}{(\Delta x)^5}.$$

If Δx be made infinitely small, the value of n remaining unaltered, what does this expression become?

(227).—Express the second difference of the product of two functions in terms of the separate functions and their respective differences, that is, show that

$$\Delta^2(u_x v_x)=u_x\Delta^2 v_x+2\Delta u_x(\Delta v_x+\Delta^2 v_x)+\Delta^2 u_x(v_x+2\Delta v_x+\Delta^2 v_x),$$

and by means of the result, find $\Delta^2(x\log x)$.

(228).—If u_x be a function of x of the form $u_x=b_1x+b_2x^2+$&c., *ad inf.*, show that it can also be expressed in the form

$$u_x=\frac{b_1x}{1-x}+\frac{\Delta b_1x^2}{(1-x)^2}+\frac{\Delta^2 b_1x^3}{(1-x)^3}+\ldots\ldots$$

(three orders of differences will suffice).

(229).—Show that

$$u_n=\{u_{n-1}+\Delta^1u_{n-2}+\Delta^2u_{n-3}+\Delta^3u_{n-4}+\ldots+\Delta^{n-2}u_1\}+\Delta^{n-1}u_1,$$

and hence determine a series of such a nature that the terms after the first shall be respectively double the first terms of the successive orders of differences ($u_2=2\Delta^1u_1$, $u_3=2\Delta^2u_1$, and so on).

CHAPTER XXIII.—INTERPOLATION.

(230).—Investigate an expression for $\Delta^n u_x$ in terms of u_x and its successive values. Using the formula thus found, if in the series 1, 6, 21, 56, κ, 252, 462, &c., the sixth differences vanish, find κ, and the sum of the series to 10 terms.

(231).—Given u_0, u_1, u_2, u_3, u_4, and u_5, and assuming fifth differences to be constant, show that

$$u_{2\frac{1}{2}}=\frac{c}{2}+\frac{25(c-b)+3(a-c)}{256}$$

u_0+u_5, $b=u_1+u_4$, $c=u_2+u_3$.

Given $u_0=100{,}000$, $u_4=98{,}391$, $u_5=98{,}011$, $u_6=97{,}615$, the series from u_{10} to u_{15}, assuming that third differences are

If $u_0=100{,}000$, $u_7=97{,}624$, $u_8=97{,}245$, and $u_9=96{,}779$, find inclusive by means of constant *third* differences.

(234).—The H^{M} premium at age 40 is at 3 per-cent $=$ ·025891

„ „ „ $3\frac{1}{2}$ „ $=$ ·024654

„ „ „ 4 „ $=$ ·023517

„ „ „ $4\frac{1}{2}$ „ $=$ ·022470

„ „ „ 5 „ $=$ ·021509

„ „ „ 6 „ $=$ ·019811

Interpolate the corresponding at $5\frac{1}{2}$ per-cent—(α) using *two* of these values; (β) using *four*; and (γ) using *six*.

(235).—Having given $\log 50 = 1·698970$

$\log 52 = 1·716003$

$\log 54 = 1·732394$

$\log 55 = 1·740363$

find, as accurately as possible from the above data, the value of log 53.

(236).—Find u_{12} and also u_2, when $u_5 = 55$, $u_6 = 126$, $u_7 = 259$, $u_8 = 484$, $u_9 = 837$, and Δ^4 is constant.

(237).—Given $\log 235 = 2·3710679$

$\log 236 = 2·3729120$

$\log 237 = 2·3747483$

$\log 238 = 2·3765770$,

find log 23563.

(238).—Find the Northampton 3 per-cent annuity for age 30, from the following table:—

Age	Northampton 3 per-cent Annuity
21	18·4708
25	17·8144
29	17·1070
33	16·3432
37	15·5154

(239).—(α) Given every *n*th term of a series of values, *i.e.*, u_x, u_{x+n}, u_{x+2n}, &c., show at length how the intermediate terms u_{x+1}, u_{x+2}, &c., may be obtained by interpolation.

(β) Given that in the series u_x, u_{x+1}, u_{x+2}, ...

$u_x = 9936675·4$

$\delta_1 = +12767·62$

$\delta_2 = -\ 3013·725$

$\delta_3 = +\ 422·8247$

$\delta_4 = -\ 34·72847$

$\delta_5 = +\ 1·254221$

you are required to construct the series as far as the term u_{x+10}. What assumption is necessary?

(240).—If u_x be a function whose differences, when the increment of x is unity, are denoted by δu_x, $\delta^2 u_x \ldots\ldots$, and by Δu_x, $\Delta^2 u_x \ldots\ldots$, when the increment of x is n; then if $\delta^2 u_x$, $\delta^2 u_{x+1} \ldots\ldots$, are in geometrical progression, with common ratio q, show that

$$\frac{\Delta u_x - n\delta u_x}{(q^n-1)-n(q-1)} = \frac{\delta^2 u_x}{(q-1)^2}.$$

(241).—Having given the values of annuities at the following rates of interest, namely, at

3 per-cent $=15{\cdot}863$
$3\frac{1}{2}$ „ $=14{\cdot}941$
4 „ $=14{\cdot}105$
$4\frac{1}{2}$ „ $=13{\cdot}343$
5 „ $=12{\cdot}648$

find the value at 4·328 per-cent.

(242).—Define a "differential coefficient", and find expressions for the values of $\frac{du_x}{dx}$, $\frac{d^2u_x}{dx^2}$, $\frac{d^3u_x}{dx^3} \ldots\ldots$, in terms of the successive finite differences δu_x, $\delta^2 u_x$, $\delta^3 u_x \ldots\ldots$

(243).—Show that when fourth differences are constant

$$\frac{d^3u_x}{dx^3} = \Delta^3 u_{x-\frac{3}{2}}.$$

(244).—Prove that

$$(u_x+\delta) = \frac{D_{x-1}-D_{x+1}}{2D_x} \text{ approximately.}$$

CHAPTER XXIV.—SUMMATION.

(245).—Show that finite integration is equivalent to the summation of an infinite number of infinitely small terms.

(246).—Explain the meaning of, and the necessity for, the introduction tant in the process of integration.

—Show that $\Sigma u_{x+n} = (\Delta^{-1})(1+\Delta)^n u_x$, and thence deduce the r the sum of n terms of a series,

$$S_n u_x = nu_x + \frac{n(n-1)}{2}\Delta u_x + \ldots$$

(248).—If $u_x = c_0 x^0 + c_1 x^1 + c_2 x^2 + \ldots$, show that

$$S_n u_x = nc_0 + \frac{n(n+1)}{2} c_1 + \frac{n(n+1)(2n+1)}{6} c_2$$
$$+ \frac{n^2(n+1)^2}{4} c_3 + \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} c_4 + \ldots$$

(249).—Let $u_1, u_2, u_3, \ldots$ denote a series of quantities, and let S_x denote the sum of the first x of them (S_0 being $= 0$). Having given the values of $S_n, S_{2n}, \ldots S_{rn}$, show how to find the values of $u_1, u_2, u_3, \ldots u_{rn}$.

(250).—Find an expression for u_x in terms of ascending powers of x, where $S_5 u_1 = 1{,}365$, $S_{10} u_1 = 5{,}155$, $S_{15} u_1 = 13{,}370$, and $S_{20} u_1 = 28{,}635$; Δ^3 being constant.

(251).—(*a*) Assuming the formula for the force of mortality,

$$\mu_x = -\frac{1}{l_x} \cdot \frac{dl_x}{dx}, \text{ find } l_x \text{ in terms of } \mu_x.$$

(*β*) How can you tell, by mere inspection, that there is a mistake in the following approximate formula for the value of μ_x?

$$\mu_x = \frac{8(d_{x-1} + d_x) - (d_{x-2} + d_{x+1})}{12 l_x}.$$

What is the correct expression?

(252).—State Lubbock's formula for approximate summation, and explain generally the process by which it may be deduced. What are the advantages and disadvantages of this formula, and the limits within which it may usefully be applied, in the computation of life benefits?

(253).—Show that Σu_x may be developed in a series proceeding by the differential coefficients of u_x, and prove that no differential coefficient of an even order will appear in the expansion.

(254).—Deduce Woolhouse's formula for approximate summation

$$\Sigma^{(1)} u = \int_0^\infty u_x dx + \frac{1}{2} u_0 - \frac{1}{12} \frac{du_0}{dx} + \frac{1}{720} \frac{d^3 u_0}{dx^3} - \&c.,$$

by the method of separation of symbols.

(255).—Write down Woolhouse's formulas for approximate summation (*a*) to deduce the value of the continuous benefit from that of the yearly benefit; (*β*) to deduce the value of the benefit at intervals of $\frac{1}{m}$th from that of the yearly benefit; (γ) to deduce the value of the yearly benefit from that of the benefit at intervals of n years.

(256).—Prove the identity of Lubbock's and Woolhouse's formulas for approximate evaluation of the yearly benefit in terms of the value of the benefit at intervals of n years.

(257).—State some of the formulas of approximate summation suggested by G. F. Hardy for the calculation of benefits. Which would you select for practical purposes, in order to obtain a sufficiently close approximation without excessive labour?

(258).—Apply Lubbock's formula for approximate summation to the computation of the value of a life annuity payable half-yearly, a table of the commutation column D_x being given for all integral values of x.

(259).—Apply Woolhouse's formula of approximate summation to the computation of the value of a continuous annuity on x ($\bar{a}_x$), deduced from the value of a yearly annuity (a_x).

(260).—Calculate by Lubbock's formula of approximate summation the values of $\bar{a}_{20.30.40}$ and $\overline{A}_{20.30.40}$ according to the Life Table printed in the *Text-Book*.

(261).—Find the value of the following benefits, by the use of one of G. F. Hardy's formulas of approximate summation:—

$$\overline{A}^{1}_{20:40:60}, \quad \overline{A}^{2}_{20:\underset{1}{40}:60}, \quad \overline{A}^{2}_{20:40:60}, \quad \overline{A}^{3}_{20:40:60},$$

$$\overline{A}^{\;\;1}_{\overline{20:40}:60}, \quad \overline{A}^{1}_{20:\overline{40:60}}, \quad \overline{A}_{\overline{20:40}:\underset{3}{60}}.$$

(262).—Find the value of a reversionary life annuity to the survivor of (30) and (40) after the death of (20) $= a_{20\,\overline{30:40}}$, using one of the formulas of approximate summation.

(263).—Calculate by Woolhouse's formula of approximate summation the value of an annuity to a life aged 40, to commence on the failure of a life aged 50, provided a life aged 30 be then alive ($a_{30.\overset{1}{50}|40}$).

(264).—Calculate the value of $a_{30:\overset{2}{50}|40}$ by a formula of approximate summation.

(265).—Obtain the annual premium for an assurance payable on the death of (x), provided that event happen before the failure of the survivor of two lives (y), (z), or within t years after such failure.

--For further practical examples and illustrations of the subject of this student is referred to *Text-Book*, Chap. xii, §§ [53]–[63]; xiii, xiv, §§ [36]–[44]; xv, §§ [10], [11], [17]–[46]; also to *Journal of Actuaries*, vol. xxiv, p. 95; xxvi, 276; xxvii, 122.

SOLUTIONS.

Part I.—Interest (including Annuities-Certain).

Chapter I.

(1).—*Text-Book*, § [1]; (*J.I.A.*, vol. iii, pp. 335–338; iv, pp. 61, 72, 243, 253).

(2).—§ [6].

(3).—§ [7]. The *force of interest* or *force of discount* is the rate per annum at which each unit of capital is momently increasing by the operation of interest. It is usually denoted by δ. Thus, if $\frac{1}{m}$ is a small fraction of a year, a capital of 1 will become $1+\frac{\delta}{m}$ at the end of such interval: and, at the end of $\frac{2}{m}$ of a year, $\left(1+\frac{\delta}{m}\right)^2$; and, at the end of the year, $\left(1+\frac{\delta}{m}\right)^m$, the value of which, when m is indefinitely increased, becomes $\epsilon^\delta=1+i$; where i is the effective rate of interest: therefore

$$\delta=\log_\epsilon(1+i).$$

$$i=\epsilon^\delta-1.$$

(4).—§ [8].

(5).—Let x be the nominal annual rate of interest convertible quarterly, and i be the corresponding effective yearly rate: then the present values of a sum due three, six, and nine months hence are, respectively, in terms of the nominal rate x,

$$\left(1+\frac{x}{4}\right)^{-1}, \quad \left(1+\frac{x}{4}\right)^{-2}, \quad \left(1+\frac{x}{4}\right).$$

or in terms of the effective rate i,

$$(1+i)^{-\frac{1}{4}}, \quad (1+i)^{-\frac{1}{2}}, \quad (1+i)^{-\frac{3}{4}},$$

the relation between the nominal and effective rates being exhibited by the equation

$$\left(1+\frac{x}{4}\right)^4=(1+i).$$

(6).— $$(1{\cdot}01)^n=10$$

$$n\lambda_\epsilon 1{\cdot}01=\lambda_\epsilon 10=2{\cdot}3026$$

$$\therefore \quad n=\frac{2{\cdot}3026}{\lambda_\epsilon 1{\cdot}01}=\frac{2{\cdot}3026}{{\cdot}01-{\cdot}00005+\ldots}=231\tfrac{2}{5} \text{ approximately.}$$

(7).—§§ [9], [11]. $$\text{L}t\left(1+\frac{i}{m}\right)^{-mn}_{m=\infty}=\epsilon^{-in}$$

$$\text{L}t\left(1+\frac{i}{m}\right)^{mn}_{m=\infty}=\epsilon^{in}.$$

(8).—§ [12]. $$(\alpha) \quad \left(1+\frac{{\cdot}05}{4}\right)^{10\times4}=(1{\cdot}0125)^{40}$$

$$(\beta) \quad \left(1+\frac{{\cdot}025}{2}\right)^{-(15\times2)}=(1{\cdot}0125)^{-30}.$$

(9).—§ [15].

(10).—*Theory of Finance*, Chap. i, § (23).

(11).— Value of bill given by A to B $=v^m a$.

Value of bill given by B to A $=v^n b$.

Let x be the amount of the bill due p years hence, then

$$v^n b+v^p . x=v^m a$$

whence $$x=\frac{v^m a-v^n b}{v^p}=v^{m-p} . a-v^{n-p} . b$$

An approximate value of x may be obtained as follows:

$$(1+i)^{-n}b+(1+i)^{-p}x=(1+i)^{-m}a,$$

ximately,

$$b(1-in)+x(1-ip)=a(1-im),$$

$$x=\frac{a(1-im)-b(1-in)}{1-ip} \quad \text{approximately.}$$

Chapter II (i).

(12).— $$\text{Discount (D)} = s(1-v^n)$$

$$\text{Annuity } (a_{\overline{n}|}) = \frac{s(1-v^n)}{i}.$$

whence $$\text{D} : a_{\overline{n}|} = 1 : \frac{1}{i}$$

$$a_{\overline{n}|} : \text{D} = 1 : i.$$

(13).— $$729 \times \frac{1-(1{\cdot}05)^{-25}}{{\cdot}05}$$

$$\doteq 14{,}580\left[1-\left(\frac{3\times 7}{2\times 10}\right)^{-25}\right]$$

$$\doteq \pounds 10{,}274.\ 9s.\ 8d.$$

(14).—*At the epoch of the first payment,* three months hence, the value of the annuity as modified would be

$$=(1+a_{\overline{24}|})\times 729;$$

while the value of the original annuity *at the same date* would be

$$=v^{\frac{1}{4}}(1+a_{\overline{24}|})\times 729.$$

The difference of these two values (to be deducted from the first payment) would be

$$(1-v^{\frac{1}{4}})(1+a_{\overline{24}|})\times 729,$$

or $$[(1+i)-(1+i)^{\frac{3}{4}}]a_{\overline{25}|}\times 729 = \pounds 387.\ 12s.\ 7d.$$

(15).—§ [25].

(16).—Accumulated capital =

$$a_{\overline{n}|}(1+i)^m = \frac{v^{-m}-v^{n-m}}{i}.$$

Accumulated payments =

$$\frac{(1+i)^m-1}{i} = \frac{v^{-m}-1}{i}.$$

Difference $$=\frac{1-v^{n-m}}{i} = a_{\overline{n-m}|}$$

which has been proved in Example (15) to be the amount unpaid at the end of m years.

(17).— $50{,}000(1-P_{\overline{25}|}s_{\overline{20}|})=50{,}000\frac{a_{\overline{5}|}}{a_{\overline{25}|}}\doteq£15{,}359.\ 7s.\ 5d.$

(18).—§§ [24]-[26]. (*J.I.A.*, vol. xi, p. 172.)

(19).—§ [27]. For each unit invested the annual payment is

$$=P_{\overline{n}|}+i'=\frac{1}{s_{\overline{n}|}}+i'=\frac{1}{a_{\overline{n}|}}+(i'-i).$$

The annual payment in respect of a capital of V is thus

$$=V(P_{\overline{n}|}+i')=V\left[\frac{1}{a_{\overline{n}|}}+(i'-i)\right],$$

where $P_{\overline{n}|}$, $s_{\overline{n}|}$, and $a_{\overline{n}|}$ are calculated at the rate i.

The second formula enables us to ascertain the value of the annual payment by means of an ordinary table of "the annuity which 1 will purchase" at the rate i. *Theory of Finance*, Chap. ii, §§ (41), (42).

(20).—§ [27].

(21).—§ [28].

$$a_{\overline{n}|}=a_{\infty}-{}_{n|}a_{\infty}=\frac{1}{i}-v^n.\frac{1}{i}=\frac{1-v^n}{i}.$$

(22).— $$v^{m-n}.\frac{1-v^n}{i}=v^m s_{\overline{n}|}.$$

(23).— $${}_{d|}a_{\overline{n}|}=v^d\frac{1-v^n}{i}=V\ \text{(say)},$$

then $$\frac{iV}{v^d}=1-v^n,$$

whence $$n=\frac{\lambda\left(1-\frac{iV}{v^d}\right)}{\lambda v}.$$

(24).—Let x, y, z be the number of years during which A, B, and C may respectively enjoy the annuity: then

$$x=\frac{\lambda\cdot75}{\lambda v},\quad y=\frac{\lambda\cdot6}{\lambda v},\quad z=\frac{\lambda\cdot5}{\lambda v}.$$

CHAPTER II (ii).

(25).—§ [33].

(26).—§ [36]. (*J.I.A.*, xv, 437.)

(27).—§ [34]. Let x be the nominal and i the effective rate of interest: then the capital repaid in the tth year is, in the case of the annuity with yearly payments, $=v^{n-t+1}$, and in the case of the annuity with $m$thly payments, $=v^{n-t+1}\times\frac{i}{x}$.

(28).—§ [35]. $a^{(m)}_{\overline{n}|}=\frac{i}{x}a_{\overline{n}|}$, $\quad \bar{a}_{\overline{n}|}=\frac{i}{\delta}a_{\overline{n}|}$.

(29).—§ [38]. The required condition is, that the payments of the perpetuity are made as often as interest is convertible. See also Chap. v, pp. 119, 120.

(30).—§ [42]. (Numerical illustration of § [34].)

$$\text{Value of Annuity} \quad =\frac{1-(1\cdot025)^{-10}}{\cdot05}.$$

Amount redeemed in the $(p+1)$th instalment $=\frac{1}{2}v^{5-\frac{p}{2}}$, where $v=(1\cdot025)^{-2}$;

or $\quad =\frac{1}{2}v^{10-p}$, where $v=(1\cdot025)^{-1}$.

(31).—Equal half-yearly charge $=\frac{5{,}000}{a_{\overline{60}|}}$, where $a_{\overline{60}|}$ is computed at $2\frac{1}{2}$ per-cent.

Sinking fund $=5{,}000\text{P}_{\overline{60}|}$ (computed at $2\frac{1}{2}$ per-cent).

Amount repaid in $(p+1)$th instalment $=5{,}000\text{P}_{\overline{60}|}(1+\cdot025)^{p}$.

CHAPTER II (iii)

(32).—§ [41].

(33).—[1]Table showing the formulas for the present value of an annuity-certain for n years, instalment payable k, and interest convertible m times a year.

(α) In terms of the nominal rate of interest x (or δ, when m or $k=\infty$).

(β) In terms of the corresponding effective rate of interest i.

	$k=k$	$k=1$	$k=\infty$
$m=m$	(α) $\frac{1}{k}\cdot\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\left(1+\frac{x}{m}\right)^{\frac{m}{k}}-1}$ (β) $\frac{1}{k}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{k}}-1}$	(α) $\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\left(1+\frac{x}{m}\right)^{m}-1}$ (β) $\frac{1-(1+i)^{-n}}{i}$	(α) $\frac{1-\left(1+\frac{x}{m}\right)^{-mn}}{\delta}$ (β) $\frac{1-(1+i)^{-n}}{\lambda_\epsilon(1+i)}$
$m=1$	(α), (β) (Here $x=i$) $\frac{1}{k}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{k}}-1}$	(α), (β) (Here $x=i$) $\frac{1-(1+i)^{-n}}{i}$	(α) $\frac{1-\epsilon^{-n\delta}}{\delta}$ (β) $\frac{1-(1+i)^{-n}}{\lambda_\epsilon(1+i)}$
$m=\infty$	(α) $\frac{1}{k}\cdot\frac{1-\epsilon^{-n\delta}}{\epsilon^{\frac{\delta}{k}}-1}$ (β) $\frac{1}{k}\cdot\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{k}}-1}$	(α) $\frac{1-\epsilon^{-n\delta}}{\epsilon^{\delta}-1}$ (β) $\frac{1-(1+i)^{-n}}{i}$	(α) $\frac{1-\epsilon^{-n\delta}}{\delta}$ (β) $\frac{1-(1+i)^{-n}}{\lambda_\epsilon(1+i)}$

In all cases,

$$\frac{\text{discount on 1 for } n \text{ years,}}{\text{instalments of annuity per annum} \times \text{interest on 1 for each instalment perio}}$$

Finance, Chap. ii, § (20).

(34).—(*a*) Here $\delta=$ the nominal rate of interest convertible momently.

(*β*) Here $i=$ the nominal rate of interest convertible momently.

(*γ*) Here the annuity is payable yearly, and interest convertible momently, and $\delta=$ the nominal rate of interest; while $i=$ the effective rate of interest.

(35).— $$\frac{X}{10}\cdot\frac{1-\left(1+\frac{i}{2}\right)^{-2n}}{\left(1+\frac{i}{2}\right)^{2}-1}=X.$$

Let $\left(1+\frac{i}{2}\right)^{2}=(1+i')$; then, dividing by X,

$$\frac{1-(1+i')^{-n}}{10i'}=1,$$

whence $$n=\frac{-\lambda(1-10i')}{\lambda(1+i')}.$$

CHAPTER III.

[In the following Solutions, the symbol $t_{\overline{n}|\overline{r}|}$ is used to represent the *n*th term of the *r*th order of figurate numbers, and the symbol $a_{\overline{n}|\overline{r}|}$ is used to represent the present value of an annuity for *n* years, whose successive payments are the terms of the *r*th order of figurate numbers].

(36).— $$t_{\overline{x}|\overline{r}|}=\frac{\overline{x-1}.\overline{x-2}\ldots\ldots\overline{x-r+1}}{\underline{|r-1}}.\quad\S\,[45],\ \text{p. }68.$$

The value of this when $x<r=0$, and giving to x the values r, $r+1$, $r+2$, &c., the successive payments of the annuity become

$$\frac{\overline{r-1}.\overline{r-2}\ldots\ldots1}{\underline{|r-1}}=1$$

$$\frac{r.\overline{r-1}\ldots\ldots2}{\underline{|r-1}}=r$$

$$\frac{\overline{r+1}.r.\overline{r-1}\ldots\ldots3}{\underline{|r-1}}=\frac{\overline{r+1}.r}{\underline{|2}}$$

&c. &c.

$$\therefore\ a_{\overline{n}|\overline{r}|} = v^r + rv^{r+1} + \frac{\overline{r+1}.r}{\lfloor 2}v^{r+2} + \ldots\ldots + \frac{\overline{n-1}.\overline{n-2}\ldots\ldots r}{\lfloor n-r}v^n$$

$$= v^r(1-v)^{-r} = \left(\frac{v}{1-v}\right)^r$$

the expansion stopping at v^n.

(37).—(*a*) From first principles

$$\begin{aligned} & v + 1{\cdot}2v^2 + 1{\cdot}4v^3 + \ldots\ldots + 10v^{46} \\ = {} & v + v^2 \quad + v^3 \quad + \ldots\ldots + v^{46} \\ & \quad + {\cdot}2v^2 + {\cdot}2v^3 + \ldots\ldots + {\cdot}2v^{46} \\ & \qquad\qquad + {\cdot}2v^3 + \ldots\ldots + {\cdot}2v^{46} \\ & \qquad \ldots\ldots\ldots\ldots\ldots \\ & \qquad\qquad\qquad\qquad + {\cdot}2v^{46} \end{aligned}$$

$$= a_{\overline{46}|} + {\cdot}2\left[\frac{v(1-v^{45})}{i} + \frac{v^2(1-v^{44})}{i} + \ldots + \frac{v^{46}(1-v^0)}{i}\right]$$

$$= a_{\overline{46}|} + {\cdot}2\,\frac{a_{\overline{46}|} - 46v^{46}}{i}.$$

(*b*) By Makeham's formula (*J.I.A.*, xiv, 189),

$$a_{\overline{46}|} + {\cdot}2a_{\overline{46}|\overline{2}|} = a_{\overline{46}|} + {\cdot}2\,\frac{a_{\overline{46}|} - 46v^{46}}{i}.$$

(*c*) By Gray's formula (*J.I.A.*, xiv, 91, 182, 397),

$$\frac{1}{i} + \frac{{\cdot}2}{i^2} - v^{46}\left(\frac{10{\cdot}2}{i} + \frac{{\cdot}2}{i^2}\right)$$

$$= a_{\overline{46}|} + {\cdot}2\,\frac{a_{\overline{46}|} - 46v^{46}}{i}.$$

(38).— $$a_{\overline{21}|} + {\cdot}05\,\frac{a_{21|} - 21v^{21}}{i}.$$

(39).— $$a_{\overline{n}|} + \Delta a_{\overline{n}|\overline{2}|} + \Delta^2 a_{\overline{n}|\overline{3}|} + \Delta^3 a_{\overline{n}|\overline{4}|}$$

$$= a_{\overline{n}|} + 7a_{\overline{n}|\overline{2}|} + 12a_{\overline{n}|\overline{3}|} + 6a_{\overline{n}|\overline{4}|}$$

§§ [43]–[45].

$a_{\overline{\infty}|\overline{r}|} = \frac{1}{i^r}$. *Theory of Finance*, Chap. iii, §§ (20)–(22).

(42).—To employ Gray's formula, the values of u_{41}, Δu_{41}, $\Delta^2 u_{41}$ . . . . . must first be ascertained.

By the following method, the general law of the series and its differences for any value of m can readily be ascertained.

We have

$$u_m = u_1 + (m-1)\Delta u_1 + \frac{(m-1)(m-2)}{\underline{|2}}\Delta^2 u_1 + \frac{(m-1)(m-2)(m-3)}{\underline{|3}}\Delta^3 u_1 + \ldots$$

$$\Delta u_m = \Delta u_1 + (m-1)\Delta^2 u_1 + \frac{(m-1)(m-2)}{\underline{|2}}\Delta^3 u_1 + \ldots$$

$$\Delta^2 u_m = \Delta^2 u_1 + (m-1)\Delta^3 u_1 + \ldots$$

$$\Delta^3 u_m = \Delta^3 u_1 + \ldots$$

Inserting the values of u_1, Δu_1, $\Delta^2 u_1$, $\Delta^3 u_1$ . . . and reducing, we have

$$u_m = m^3 - 6m^2 + 13m - 7.$$

(This formula exhibits the *general law* of the series)

$$\Delta u_m = 3m^2 - 9m + 8$$

$$\Delta^2 u_m = 6(m-1)$$

$$\Delta^3 u_m = 6.$$

Or for $m = 41$

$$u_{41} = 59361 \qquad \Delta u_{41} = 4682$$

$$\Delta^2 u_{41} = 240 \qquad \Delta^3 u_{41} = 6.$$

Now, applying Gray's formula, we have, for the value of the variable annuity,

$$\frac{1}{i} + \frac{2}{i^2} + \frac{6}{i^4} - v^{40}\left(\frac{59361}{i} + \frac{4682}{i^2} + \frac{240}{i^3} + \frac{6}{i^4}\right)$$

$$= \text{(when } i = \cdot 05\text{) } 117066.$$

$$\begin{array}{llllllll}
(43).\text{—}u = & 55 & 126 & 259 & 484 & 837 & \ldots & \\
\Delta = & \ldots & 71 & 133 & 225 & 353 & \ldots & \\
\Delta^2 = & \ldots & \ldots & 62 & 92 & 128 & \ldots & \\
\Delta^3 = & \ldots & \ldots & \ldots & 30 & 36 & \ldots & \\
\Delta^4 = & \ldots & \ldots & \ldots & \ldots & 6 & \ldots &
\end{array}$$

Value

$$=55a_{\overline{40}|}+71a_{\overline{40}|\overline{2}|}+62a_{\overline{40}|\overline{3}|}+30a_{\overline{40}|\overline{4}|}+6a_{\overline{40}|\overline{5}|}$$

= (at 5 per-cent) £1,521,443.

(44).—$201a_{\overline{15}|}+102a_{\overline{15}|\overline{2}|}+38a_{\overline{15}|\overline{3}|}+9a_{\overline{15}|\overline{4}|}+a_{\overline{15}|\overline{5}|}$.

CHAPTER IV (i).

(45).—(A) ·052310; (B) ·052310; (C) ·052310; (D) ·052310.

(46).—(D_1) ·032167; (D_2) ·032167.

(47).—$a_{\overline{n}|}=\frac{1-v^n}{i}$. The nearer n approaches to ∞, the more nearly does $a_{\overline{n}|}$ approximate to $a_\infty=\frac{1}{i}$, $\therefore$ when n is large $a_{\overline{n}|}=\frac{1}{i}$ approximately, and $i=\frac{1}{a_{\overline{n}|}}$ approximately. Inserting this value in the formula $a_{\overline{n}|}=\frac{1-(1+i)^{-n}}{i}$ we have

$$a_{\overline{n}|}=\frac{1-\left(1+\frac{1}{a_{\overline{n}|}}\right)^{-n}}{i} \text{ approximately,}$$

$$\text{and } i=\frac{1}{a_{\overline{n}|}}-\frac{1}{a_{\overline{n}|}}\left(1+\frac{1}{a_{\overline{n}|}}\right)^{-n}.$$

(*Text-Book*, Example (5), p. 171.)

For the case $a_{\overline{94}|}=27$, we have

$$i=\frac{1}{27}-\frac{1}{27}\left(1+\frac{1}{27}\right)^{-94}=\cdot03582 \text{ approximately.}$$

(48).—§§ [54]–[63].

CHAPTER IV (ii).

(49).—§§ [66], [67]. (*J.I.A.*, xviii, 132.)

(50).—Here the formula

$$C'+(C-C')\frac{i}{j}, \quad (\S\ [67])$$

$$v'^n+(1-v'^n)\frac{i}{j}, \quad \text{where } v'^n=(1+j)^{-n},$$

$$1-a'_{\overline{n}|}(j-i), \quad \text{where } a'_{\overline{n}|}=\frac{1-(1+j)^{-n}}{j}.$$

(51).—§ [69].

(52.)— $$\frac{p}{100}\cdot\frac{(1+i')^n-1}{i'}=1,$$

whence $$n=\frac{\lambda\left(\frac{100i'}{p}+1\right)}{\lambda(1+i')},$$ which is independent of the rate i.

(*Text-Book*, Example (28).)

(53).—By Makeham's formula, we have

$$\left(1+\frac{p}{100}\right)=v'^n+(1-v'^n)\frac{i}{i'},$$

whence $$i'=\frac{(1-v'^n)i}{\left(1+\frac{p}{100}\right)-v'^n}.$$

Insert a value of v'^n near to the true rate, and then deduce i' by successive approximations.

Or, as follows:—

$$a'_{\overline{n}|}i+v'^n=1+\frac{p}{100}$$

but $$a'_{\overline{n}|}i'+v'^n=1$$

$$\therefore\quad a'_{\overline{n}|}(i-i')=\frac{p}{100}$$

$$i-i'=\frac{p}{100a'_{\overline{n}|}}$$

and $$i'=i-\frac{p}{100a'_{\overline{n}|}},$$

from which i' may be obtained by successive approximations.

(54).—(*Text-Book*, Example (6), p. 134).

Here we have

$$p=[i-i'(1+p)]\frac{(1+j)^n-1}{j}$$

Let $$\frac{(1+j)^n-1}{j}=\frac{1}{P_{\overline{n}|}},$$

then $$p=[i-i'(1+p)]\frac{1}{P_{\overline{n}|}}$$

$$=\frac{i-i'}{P_{\overline{n}|}+i'}.$$

(55).—(α) $1{,}000\left\{v'^{20}+(1-v'^{20})\frac{\cdot 03}{\cdot 05}\right\}=1{,}000\{1-a'_{\overline{20}|}(\cdot 05-\cdot 03)\}$

$$=\pounds 750.\ 15s.\ 1d.$$

(β) $1{,}000\left\{\frac{a'_{\overline{20}|}}{20}+\left(1-\frac{a'_{\overline{20}|}}{20}\right)\frac{\cdot 03}{\cdot 05}\right\}=1{,}000\left\{1-\left(1-\frac{a'_{\overline{20}|}}{20}\right)\left(1-\frac{\cdot 03}{\cdot 05}\right)\right\}$

$$=\pounds 849.\ 4s.\ 11d.$$

(56).—By Makeham's formula

$$A=C'+(C-C')\frac{i}{j'},$$

we have

$$M\left[\frac{a'_{\overline{n}|}}{n}+\left(1-\frac{a'_{\overline{n}|}}{n}\right)\frac{i}{j}\right]=M\left[1-\left(1-\frac{a'_{\overline{n}|}}{n}\right)\left(1-\frac{i}{j}\right)\right],$$

where $a'_{\overline{n}|}$ is computed at the rate j.

The problem may also be solved as follows:

The successive payments are

$$\frac{M}{n}+iM \qquad =\frac{M}{n}(1+in)$$

$$\frac{M}{n}+iM\left(\frac{n-1}{n}\right)=\frac{M}{n}(1+i.\overline{n-1})$$

$$\frac{M}{n}+iM\left(\frac{n-2}{n}\right)=\frac{M}{n}(1+i.\overline{n-2})$$

. . . .

$$\frac{M}{n}+iM\left(\frac{1}{n}\right) \qquad =\frac{M}{n}(1+i)$$

The present value of these payments at the rate j is

$$=\frac{M}{n}[a'_{\overline{n}|}+i(a'_{\overline{n}|}+a'_{\overline{n-1}|}+\ldots\ldots+a'_{\overline{1}|})]$$

$$=\frac{M}{n}\left[a'_{\overline{n}|}+\frac{i}{j}(1-v'^{n}+1-v'^{n-1}+\ldots\ldots+1-v')\right]$$

$$=\frac{M}{n}\left[a'_{\overline{n}|}+\frac{i}{j}(n-a'_{\overline{n}|})\right]=M\left[\frac{a'_{\overline{n}|}}{n}+\left(1-\frac{a'_{\overline{n}|}}{n}\right)\frac{i}{j}\right]$$

(This example illustrates the advantages of Makeham's dealing with similar problems.)

(57).—Let P be the amount of the property at the commencement, and $P_1, P_2, P_3, \ldots\ldots$ the amounts at the end of the 1st, 2nd, 3rd, ... years. Then

$$P_1=P[1+(1-m)i]$$
$$P_2=P_1[1+(1-2m)i]$$
$$P_3=P_2[1+(1-3m)i]$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$P_p=P_{p-1}[1+(1-pm)i]$$
$$P_{2p}=P_{2p-1}[1+(1-2pm)i]=0.$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

Therefore $$1+(1-2pm)i=0\,;$$

whence $$1+i=2pmi.$$

Amount spent in pth year

$$=pmiP_{p-1}=\frac{1+i}{2}P_{p-1}.$$

Amount left at end of pth year

$$=P_p=P_{p-1}[1+(1-pm)i]=\frac{1+i}{2}P_{p-1}.$$

(58).— $$76=(6+1)\frac{1-(1+x)^{-n}}{x}\,;$$

whence $$10{\cdot}85714=a_{\overline{n}|} \text{ at rate } x.$$

Also $$100=1\times\frac{(1+{\cdot}06)^n-1}{{\cdot}06}\,;$$

whence $$n=33\tfrac{1}{3} \text{ approximately.}$$

Then we have $$10{\cdot}85714=a_{\overline{33\frac{1}{3}}|} \text{ at rate } x\,;$$

whence, by formula (C), § [57], $x={\cdot}086265$

CHAPTER VII.

(59), (60).—§§ [84], [85]. *Tables and Formulæ*, (Gray), Chap. ii, §§ (47)-(55), (68)-(76).

(61), (62).—*Theory of Finance*, Chap. v, §§ (28)-(30). *Tables and Formulæ*, Chap. ii, §§ (63)-(65), (74), (75) (63).—Let $\mathbf{a}_{\overline{n}|}$ be the present value of an annuity-certain for n years payable in advance, then

$$\mathbf{a}_{\overline{n}|}=a_{\overline{n-1}|}+1.$$

Similarly, let $\mathbf{s}_{\overline{n}|}$ be the amount of an annuity-certain for n years payable in advance, then

$$\mathbf{s}_{\overline{n}|}=s_{\overline{n+1}|}-1.$$

We have also

$$\mathbf{a}_{\overline{n}|}=(1+i)a_{\overline{n}|}$$

$$\mathbf{s}_{\overline{n}|}=(1+i)s_{\overline{n}|}.$$

Chap. ii, § [29].

(64).—*Theory of Finance*, Chap. v, §§ (19), (23), (26).

$$\Sigma_1^t(1+i)^n=s_{\overline{t+1}|}-1$$

$$\Sigma_1^t v^n=a_{\overline{t}|}$$

$$\Sigma_1^t s_{\overline{n}|}=\frac{(1+i)s_{\overline{t}|}-t}{i}$$

$$\Sigma_1^t a_{\overline{n}|}=\frac{t-a_{\overline{t}|}}{i}.$$

MISCELLANEOUS EXAMPLES.

(65).—Let x be the effective rate of interest, and let it be assumed that the half-yearly dividends can be immediately re-invested at the rate x: then the accumulated payments in respect of any year are approximately equal to

$$2{\cdot}5+2{\cdot}5\left(1+\frac{x}{2}\right)+3=(8+1{\cdot}25x),$$

which, by the terms of the question, $=130x$,

whence $\qquad x={\cdot}062136$ nearly.

If the dividends can be re-invested at a rate of interest of i per annum, the formula becomes

$$8+1{\cdot}25i=130x,$$

$$x=\frac{8+1{\cdot}25i}{130},$$

inserting the value of i, that of x can be obtained.

(66).—Here the accumulated payments in respect of any year, at the effective rate x (the re-investments of dividend being made at the same rate), become approximately equal to

$$2{\cdot}5(1+x)+2{\cdot}5\left(1+\frac{x}{2}\right)+3\left(1+\frac{x}{2}\right)=8+5{\cdot}25x.$$

But, in the solution to Problem (65), it was shown that

$$8+1{\cdot}25x=130x$$

$$\therefore\quad 8+5{\cdot}25x=134x,$$

and the price to be given is therefore £134.

If the re-investments of dividend are made at the rate i, we have, for the accumulated payments of any year,

$$8+5{\cdot}25i.$$

But, in the solution to Problem (65), it was shown that, in this case,

$$8+1{\cdot}25i=130x$$

$$\therefore\quad 8+5{\cdot}25i=130x+4i,$$

in which, by inserting values for x and i, the required price may be obtained.

It is also evident that, where re-investments of dividend are made at the rate x, the value of the share just before payment of the June dividend, is equal to the value just after payment of the December dividend and bonus, *plus* six months' interest thereon,

$$=130\left(1+\frac{x}{2}\right)=(130\times 1{\cdot}031068)=134 \text{ approximately.}$$

(67).—(a) Dividing the income from interest by the "mean fund" in the middle of the year, we have

$$\frac{45{,}000}{1{,}037{,}500}={\cdot}043373=£4.\ 6s.\ 9d.\ \text{per-cent.}$$

This result represents the "force of interest" (δ), and the yearly rate realized may be deduced by the usual relation,

$$\delta=\log_{\epsilon}(1+i)=\log_{10}(1+i)\times 2{\cdot}302585,$$

whence $\dfrac{{\cdot}043373}{2{\cdot}302585} = {\cdot}018837 = \log_{10}(1+i)$;

and $i = {\cdot}04433 = £4.\ 8s.\ 8d.$ per-cent.

(β) If i be the effective rate of interest, the fund of £1,000,000 becomes, at the end of the year, $= 1{,}000{,}000(1+i)$; and the income (excluding interest), less the outgo, forms a continuous annuity for one year of £30,000, the amount of which, at the end of the year,

$$= 30{,}000 \frac{i}{\log_\epsilon(1+i)} = 30{,}000\left(1 + \frac{i}{2} - \frac{i^2}{12} + \ldots\right).$$

Hence, the interest earned in the year

$$= 1{,}000{,}000\,i + 30{,}000\left(\frac{i}{2} - \frac{i^2}{12} + \ldots\right) = 45{,}000;$$

whence $i = \dfrac{45{,}000}{1{,}015{,}000} = {\cdot}04433$

$= £4.\ 8s.\ 8d.$ per-cent approximately.

(68).—Let n = the number of times interest is convertible

then $\left(1 + \dfrac{{\cdot}06}{n}\right)^n = 1{\cdot}061678$

$$\log_\epsilon\left(1 + \frac{{\cdot}06}{n}\right)^n = n\left(\frac{{\cdot}06}{n} - \frac{{\cdot}0036}{2n^2} + \frac{{\cdot}000216}{3n^3} - \ldots\right)$$

whence $\log_\epsilon 1{\cdot}061678 = 06 - \dfrac{{\cdot}0018}{n} + \dfrac{{\cdot}000072}{n^2} - \ldots$

or, ${\cdot}0598505 = {\cdot}06 - \dfrac{{\cdot}0018}{n} + \dfrac{{\cdot}000072}{n^2} - \ldots$

and ${\cdot}0001495 = \dfrac{{\cdot}0018}{n} - \dfrac{{\cdot}000072}{n^2} + \ldots$

from which it is obvious that $n=12$ nearly; and substituting $12n$ for n^2, we have

$$\begin{aligned} {\cdot}0001495 &= \frac{{\cdot}0018}{n} - \frac{{\cdot}000072}{12n} \\ &= \frac{1}{n}({\cdot}0018 - {\cdot}000006) \\ n &= \frac{{\cdot}001794}{{\cdot}0001495} = 12. \end{aligned}$$

(69).—Nominal rate $=\frac{\cdot 025}{94}=\cdot 02660=$ £2. 13*s*. 2*d*. per-cent.

Effective rate (with quarterly dividends)

$$\doteq(1\cdot 00665)^4-1=\cdot 026865=\text{£}2.\ 13s.\ 9d.\ \text{per-cent.}$$

Instantaneous rate $=\delta=\log_\epsilon(1+i)$, where $i=$ above effective rate

$$=\log_\epsilon 1\cdot 026865=\log_{10}1\cdot 026865\times 2\cdot 302585$$

$$=\cdot 026513=\text{£}2.\ 13s.\ 0d.\ \text{per-cent.}$$

(70).—We have (*Algebra*—Exponential and Logarithmic Series)

$$\log_\epsilon\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\dots\right\};$$

therefore

$$\log_\epsilon\frac{1+(n+\frac{1}{2})i}{1+(n-\frac{1}{2})i}=2\left\{\frac{i}{2+2ni}+\frac{1}{3}\left(\frac{i}{2+2ni}\right)^3+\dots\right\}.$$

If i is a small fraction this becomes approximately,

$$\log_\epsilon\frac{1+(n+\frac{1}{2})i}{1+(n-\frac{1}{2})i}=\frac{i}{1+ni}.$$

Giving to n successively the values 1, 2, 3, . . . n, we have therefore,

$$\frac{1}{1+i}+\frac{1}{1+2i}+\frac{1}{1+3i}+\dots+\frac{1}{1+ni}$$

$$=\frac{1}{i}\left\{\log_\epsilon\frac{1+1\frac{1}{2}i}{1+\frac{1}{2}i}+\log_\epsilon\frac{1+2\frac{1}{2}i}{1+1\frac{1}{2}i}+\dots+\log_\epsilon\frac{1+(n+\frac{1}{2})i}{1+(n-\frac{1}{2})i}\right\}$$

$$=\frac{1}{i}\left\{\log_\epsilon\left(\frac{1+1\frac{1}{2}i}{1+\frac{1}{2}i}\cdot\frac{1+2\frac{1}{2}i}{1+1\frac{1}{2}i}\cdot\frac{1+3\frac{1}{2}i}{1+2\frac{1}{2}i}\cdot\dots\cdot\frac{1+(n+\frac{1}{2})i}{1+(n-\frac{1}{2})i}\right)\right\}$$

$$=\frac{1}{i}\log_\epsilon\frac{1+(n+\frac{1}{2})i}{1+\frac{1}{2}i}\ \text{approximately.}$$

SOLUTIONS.

PART II.—LIFE CONTINGENCIES (INCLUDING LIFE ANNUITIES AND ASSURANCES).

CHAPTER I.

(1).—*Text-Book* §§ [1]-[4].

(2).—If out of $(m+n)$ trials, the result A has happened m times, and the result B n times, then the probability that the next trial will produce the result A is strictly $\frac{m+1}{(m+1)+(n+1)} = \frac{m+1}{m+n+2}$, or, in the present case, $\frac{30}{2002}$. (*De Morgan on Probabilities*, Chap. iii, p. 65.)

This result is, however, based upon the assumption that all values of the required probability are, *à priori*, equally likely, which cannot be said to be true with regard to the probabilities of death.

(3).—From the conditions laid down, the total deaths in any year will be equal to the number of annual births. If this $=l_0$ we can obtain by successive multiplication by p_0, p_1, p_2 . . . . ., the values of l_1, l_2, l_3 . . . . . the survivors at the several ages. If we assume the numbers living between ages 0 and 1, 1 and 2, &c., to be $\frac{l_0+l_1}{2}$, $\frac{l_1+l_2}{2}$, &c., the

e numbers will represent the total population.

§ [13].

α) § [15].

β) The effect would be to exaggerate the mortality, especially at the younger ages. (*J.I.A.*, xviii, 107.)

Chapter II.

(6).—Expressing the given formula in terms of the number living at each age, we have

$$\frac{l_{x+1}}{l_x}\cdot\frac{l_{x+2}}{l_{x+1}}\cdot\frac{l_{x+3}}{l_{x+2}}\cdots\cdots\frac{l_{x+n}}{l_{x+n-1}}=\frac{l_{x+n}}{l_x}={}_np_x.$$

(7).—We have

$$
\begin{aligned}
&{}_np_x(d_x+d_{x+1}+\ldots\ldots+d_{\omega-1})\\
&=\frac{l_{x+n}}{l_x}(l_x-l_{x+1}+l_{x+1}-l_{x+2}+\ldots\ldots+l_{\omega-2}-l_{\omega-1}+l_{\omega-1})\\
&=l_{x+n}.
\end{aligned}
$$

(8).—Proportion per-cent of married couples $=100\times\frac{2236}{2501}\times\frac{2374}{2611}$
$=81{\cdot}29.$

„ „ widowers $=100\times\frac{2236}{2501}\times\frac{237}{2611}$
$=8{\cdot}12.$

„ „ widows $=100\times\frac{265}{2501}\times\frac{2374}{2611}$
$=9{\cdot}63.$

(The remaining ·96 per-cent represent the deceased couples.)

(9).—(a) § [15]. $1-{}_np_{xy}.$

(β) § [12]. $1-(1-{}_np_x)(1-{}_np_y).$

(10).—(a) § [18]. ${}_{n-1|}q_x\times{}_{n-1|}q_y=\frac{d_{x+n-1}}{l_x}\cdot\frac{d_{y+n-1}}{l_y}.$

(β) § [21]. ${}_{n-1|}q_x(1-{}_{n-1|}q_y)+{}_{n-1|}q_y(1-{}_{n-1|}q_x)$

$$=\frac{d_{x+n-1}}{l_x}+\frac{d_{y+n-1}}{l_y}-\frac{2\,d_{x+n-1}d_{y+n-1}}{l_xl_y}.$$

(11).—§ [11]. ${}_{|n}q_{\overline{xy}}={}_{|n}q_x\times{}_{|n}q_y=\frac{l_x-l_{x+n}}{l_x}\cdot\frac{l_y-l_{y+n}}{l_y}.$

§ [20]. ${}_{n-1|}q_{\overline{xy}}={}_{|n}q_{\overline{xy}}-{}_{|n-1}q_{\overline{xy}}.$

(12).—

$${}_np_x\times{}_{n-1}p_y=\frac{l_{x+n}}{l_x}\cdot\frac{l_{y+n-1}}{l_y}=\frac{l_{x+n}}{l_{x+1}}\cdot\frac{l_{y+n-1}}{l_y}\cdot\frac{l_{x+1}}{l_x}={}_{n-1}p_{x+1:y}\times p_x,$$

also

$${}_np_x\times{}_{n-1}p_y=\frac{l_{x+n}}{l_x}\cdot\frac{l_{y+n-1}}{l_y}=\frac{l_{x+n}}{l_x}\cdot\frac{l_{y+n-1}}{l_{y-1}}\cdot\frac{l_{y-1}}{l_y}=\frac{{}_np_{x:y-1}}{p_{y-1}}.$$

(13).—$1-\{1-({}_np_x-{}_{n+m}p_x)\}\{1-({}_np_y-{}_{n+m}p_y)\}\{1-({}_np_z-{}_{n+m}p_z)\}$

(14).—If the ages of the three lives are severally x, y, z, the following are the various contingencies:—

Die.	Survive.	Probability.
None	$x.y.z$	$p_x\times p_y\times p_z$
x	$y.z$	$(1-p_x)p_y\times p_z$
y	$x.z$	$(1-p_y)p_x\times p_z$
z	$x.y$	$(1-p_z)p_x\times p_y$
xy	z	$(1-p_x)(1-p_y)p_z$
xz	y	$(1-p_x)(1-p_z)p_y$
yz	x	$(1-p_y)(1-p_z)p_x$
xyz	None	$(1-p_x)(1-p_y)(1-p_z)$

It will be found on expanding these several probabilities that the total is unity, which shows that all possible cases have been included.

(15).—(a) $1-(1-{}_{n-1|}q_x)(1-{}_{n-1|}q_y)(1-{}_{n-1|}q_z)$.

(β) $1-({}_{n-1|}q_x\times{}_{n-1|}q_y\times{}_{n-1|}q_z)$.

(γ) $(1-{}_{n-1}p_x)[({}_{n-1}p_y-{}_np_y)\,{}_np_z+({}_{n-1}p_z-{}_np_z)\,{}_np_y]$

$+(1-{}_{n-1}p_y)[({}_{n-1}p_x-{}_np_x)\,{}_np_z+({}_{n-1}p_z-{}_np_z)\,{}_np_x]$

$+(1-{}_{n-1}p_z)[({}_{n-1}p_x-{}_np_x)\,{}_np_y+({}_{n-1}p_y-{}_np_y)\,{}_np_x]$.

(16).—§§ [27], [28].

(17).—§ [34].

(18).—§ [34]. If, however, $(n+1)p$ is an integer, say κ, the rth and $(r+1)$th terms are equal, and the occurrence of either $(r-1)$ or r deaths is equally probable, and more probable than that of any other number.

Let p be the probability of any one of the lives surviving the d: then the required expression is the sum of the last four expansion of $\{p+(1-p)\}^7$, that is,

$$p^3(1-p)^4+21p^2(1-p)^5+7p(1-p)^6+(1-p)^7.$$

(21).—(a) $1-\left\{(\cdot98)^{100}+100(\cdot98)^{99}(\cdot02)+\frac{100\times99}{2}(\cdot98)^{98}(\cdot02)^{2}\right\}$.

(β) $1-\left\{(\cdot98)^{1000}+1000(\cdot98)^{999}(\cdot02)+\dots\dots\right.$

$$\left.+\frac{\underline{|1000}}{\underline{|29}\,\underline{|971}}(\cdot98)^{971}(\cdot02)^{29}\right\}.$$

(22).—Let $q=(1-p)$, represent the probability of dying during the year.

(a) $\frac{\underline{|1000}}{\underline{|20}\,\underline{|980}}p^{980}q^{20}$.

(β) $p^{1000}+1000p^{999}q+\frac{\underline{|1000}}{\underline{|2}\,\underline{|998}}p^{998}q^{2}+\dots+\frac{\underline{|1000}}{\underline{|20}\,\underline{|980}}p^{980}q^{20}$.

(γ) $q^{20}p^{980}$.

(δ) $q^{20}\times(p+q)^{980}=q^{20}$, since $(p+q)=1$.

(23).—(a) $\frac{1}{\underline{|6}}=\cdot00139$.

(β) $\frac{\underline{|4}}{\underline{|6}}=\cdot03333$.

(γ) $\frac{1-(\cdot99)^{6}}{6}=\cdot00974$.

(24).—If a number of persons are exposed to risk of death at the same age x,

The "rate of mortality" (q_x) is the ratio of those dying within a year to the number living at age x.

The "force of mortality" (μ_x) is the annual rate at which the lives are dying at age x. It may also be defined as the limit of the expression

$$\frac{l_x-l_{x+\frac{1}{m}}}{\frac{1}{m}l_x}$$

when m is indefinitely increased.

The "central death-rate" (m_x) is the ratio of the numbers dying within a year to the *average* number living during the year.

$$m_x=\frac{d_x}{l_{x+\frac{1}{2}}}=\mu_{x+\frac{1}{2}}\text{ approximately}=\frac{2q_x}{2-q_x}$$

$$q_x=\frac{2m_x}{2+m_x}=\frac{2\mu_{x+\frac{1}{2}}}{2+\mu_{x+\frac{1}{2}}}\text{ approximately.}$$

(25).— $$\mu_x >=< q_x$$

according as $\dfrac{d_{x-1}+d_x}{2l_x} >=< \dfrac{d_x}{l_x}$

„ „ $d_{x-1}+d_x >=< 2d_x$

„ „ $d_{x-1} >=< d_x$

(26).—§ [37] (*J.I.A.*, xvi, 450.) It has been shown (Ex. 24) that μ_x is equal to the limit of the expression $\dfrac{l_x - l_{x+\frac{1}{m}}}{\frac{1}{m} l_x}$ when $\dfrac{1}{m}$ vanishes. By the definition of $\dfrac{d}{dx} l_x$, it is the limit of $\dfrac{l_{x+\frac{1}{m}} - l_x}{\frac{1}{m}}$. Hence $\mu_x = -\dfrac{1}{l_x}\dfrac{d}{dx} l_x$.

(27).—This arises from the general principle of the Differential Calculus, that, where ϕ_x is any function of x,

$$\frac{1}{\phi_x} \cdot \frac{d}{dx} \cdot \phi_x = \frac{d}{dx} \log_\epsilon \phi_x .$$

(28).— $$q_x = \frac{2\mu_{x+\frac{1}{2}}}{\mu_{x+\frac{1}{2}}+2}. \quad \text{(Chap. i, § [17].)}$$

(29).—We have $$\mu_{x+t} = -\frac{\frac{d}{dt} l_{x+t}}{l_{x+t}}$$

$$\therefore\ l_{x+t} \cdot \mu_{x+t} = -\frac{d}{dt} l_{x+t},$$

and $$\int_0^1 l_{x+t} \cdot \mu_{x+t} dt = -\int_0^1 \left(\frac{d}{dt} l_{x+t}\right) dt$$
$$= l_x - l_{x+1} = d_x,$$

whence $$\frac{1}{l_x}\int_0^1 l_{x+t}\ \mu_{x+t} dt = \frac{d_x}{l_x} = q_x .$$

§ [44].

CHAPTER III.

(31).—(*a*) The “average duration of life” ($\mathring{e}_x$) is the number of years which persons of a specified age, taken one with another, survive, according to the given table of mortality.

$$\mathring{e}_x=\tfrac{1}{2}+\frac{l_{x+1}+l_{x+2}+\ldots\ldots}{l_x}$$

(*β*) The “probable lifetime” (*vie probable*) is the number of years which a person of given age has an even chance of surviving according to the experience of a particular mortality table.

$=t$ deduced from the equation $\dfrac{l_{x+t}}{l_x}=\tfrac{1}{2}$.

(*γ*) The “mean age at death” is the average age to which persons of a given age, taken one with another, will survive, according to the experience of a particular mortality table.

$=x+\mathring{e}_x$.

(32).—§§ [1]-[4], [8].

Age	Average Duration of Life. $\mathring{e}_x$	Average Age at Death. $(x+\mathring{e}_x)$	“Probable Lifetime” (t) $\frac{l_{x+t}}{l_x}=\frac{1}{2}$
80	5·19	85·19	3·75
81	5·11	86·11	4·21
82	5·23	87·23	4·00
83	5·27	88·27	4·25
84	4·95	88·95	4·20
85	4·38	89·38	3·70
86	4·58	90·58	4·25
87	4·32	91·32	3·75
88	3·84	91·84	3·17
89	3·54	92·54	3·00
90	2·98	92·98	2·50
91	2·97	93·97	2·75
92	2·58	94·58	2·50
93	2·28	95·28	2·25
94	1·79	95·79	1·75
95	1·30	96·30	1·25
96	·83	96·83	·75
97	·50	97·50	·50

(33).— $e_x = (1 - q_x)(1 + e_{x+1})$ or $\mathring{e}_x = \frac{1}{2} + (1 - q_x)(\frac{1}{2} + \mathring{e}_{x+1})$.

The values for q_x being given for all ages, and the expectation at the limiting age being *zero*, that for all younger ages could be found by successively applying one or other of the above formulas.

(34).—(α) At age $(x + r)$, where d_{x+r} is the greatest value of d from age x to the end of life.

(β) t years, where $\frac{l_{x+t}}{l_x} = \frac{1}{2}$

(35).—(*a*) § [9].

(β) The probability of a life aged x dying in the nth year $= {}_{n-1|}q_x = \frac{d_{x+n}}{l_x}$. But, by De Moivre's hypothesis,

$$d_x = d_{x+1} = \ldots\ldots = d_{x+n} = d_{x+n+1} = \&c. = 1$$

$$\therefore\ {}_{0|}q_x = {}_{1|}q_x = \ldots\ldots = {}_{n-1|}q_x = {}_{n|}q_x = \&c. = \frac{1}{l_x} \text{ for all values of } n.$$

$$(\gamma)\ \mu_x = -\frac{1}{l_x}\frac{d}{d_x}l_x \qquad l_x = 86 - x \qquad \therefore\ \frac{d}{dx}l_x = -1$$

$$\therefore\ \mu_x = \frac{1}{l_x} = \frac{d_x}{l_x} = q_x.$$

This equality also follows from the reasoning in Solution (25), bearing in mind that on De Moivre's hypothesis $d_{x-1} = d_x$ for all values of x.

(36).—$q_x + p_x(1 + q_{x+1}) + {}_2p_x(1 + q_{x+2}) + \ldots\ldots$

$$= \frac{d_x}{l_x} + \frac{l_{x+1}}{l_x} + \frac{l_{x+1}}{l_x}\cdot\frac{d_{x+1}}{l_{x+1}} + \frac{l_{x+2}}{l_x} + \frac{l_{x+2}}{l_x}\cdot\frac{d_{x+2}}{l_{x+2}} + \ldots\ldots$$

$$= \frac{1}{l_x}(d_x + l_{x+1} + d_{x+1} + l_{x+2} + d_{x+2} + \ldots\ldots)$$

$$= \frac{1}{l_x}(l_x + l_{x+1} + l_{x+2} + \ldots\ldots) = 1 + e_x$$

(37).— $${}_{n|m}e_x = \frac{l_{x+n+1} + l_{x+n+2} + \ldots\ldots + l_{x+n+m}}{l_x}$$

$$= {}_np_x(e_{x+n} - {}_mp_{x+n}e_{x+n+m})$$

$${}_{n|m}\mathring{e}_x = {}_np_x(\mathring{e}_{x+n} - {}_mp_{x+n}\mathring{e}_{x+n+m})$$

$$= {}_{n|m}e_x + \frac{1}{2}\cdot\frac{l_{x+n} - l_{x+n+m}}{l_x}.$$

§ [18].

We have $\mathring{e}_x = \frac{1}{2} + p_x(\frac{1}{2} + \mathring{e}_{x+1})$. [Example (33).]

Whence
$$p_x=\frac{\overset{\circ}{e}_x-\frac{1}{2}}{\overset{\circ}{e}_{x+1}+\frac{1}{2}}$$

$$p_{x-1}=\frac{\overset{\circ}{e}_{x-1}-\frac{1}{2}}{\overset{\circ}{e}_x+\frac{1}{2}}$$

$$p_x=\frac{\overset{\circ}{e}_x-\frac{1}{2}}{\overset{\circ}{e}_{x+1}+\frac{1}{2}}$$

$$p_{x+1}=\frac{\overset{\circ}{e}_{x+1}-\frac{1}{2}}{\overset{\circ}{e}_{x+2}+\frac{1}{2}}\,.$$

If $\overset{\circ}{e}_{x-1}=\overset{\circ}{e}_x=\overset{\circ}{e}_{x+1}=e$, the above becomes

$$p_{x-1}=\frac{e-\frac{1}{2}}{e+\frac{1}{2}}$$

$$p_x=\frac{e-\frac{1}{2}}{e+\frac{1}{2}}$$

$$\therefore\ p_{x-1}=p_x\,.$$

Also
$$\overset{\circ}{e}_{x-1}=\tfrac{1}{2}+p_{x-1}(\tfrac{1}{2}+\overset{\circ}{e}_x)$$

$$\overset{\circ}{e}_x=\tfrac{1}{2}+p_x(\tfrac{1}{2}+\overset{\circ}{e}_{x+1})$$

$$\overset{\circ}{e}_{x+1}=\tfrac{1}{2}+p_{x+1}(\tfrac{1}{2}+\overset{\circ}{e}_{x+2})\,.$$

If $p_{x-1}=p_x=p_{x+1}=p$, these become

$$\overset{\circ}{e}_{x-1}=\tfrac{1}{2}+p(\tfrac{1}{2}+\overset{\circ}{e}_x)$$

$$\overset{\circ}{e}_x=\tfrac{1}{2}+p(\tfrac{1}{2}+\overset{\circ}{e}_{x+1})$$

$$\overset{\circ}{e}_{x+1}=\tfrac{1}{2}+p(\tfrac{1}{2}+\overset{\circ}{e}_{x+2})\,.$$

Then $\overset{\circ}{e}_{x-1}=\overset{\circ}{e}_x$ only if $\overset{\circ}{e}_x=\overset{\circ}{e}_{x+1}=\overset{\circ}{e}_{x+2}$, and we have

$$\overset{\circ}{e}_{x+2}=\tfrac{1}{2}+p(\tfrac{1}{2}+\overset{\circ}{e}_{x+2}),\ \text{whence}\ \overset{\circ}{e}_{x+2}=\frac{1}{2}\cdot\frac{1+p}{1-p}\,.$$

(39).—§ [1]. Assuming that the deaths are equally distributed through the year,

mq_x persons originally aged x who die in the first year live $\frac{1}{2}mq_x$ years
$m_{1|}q_x$ " " " second " $\frac{3}{2}m_{1|}q_x$
$m_{n-1|}q_x$ " " " nth " $\frac{2n-1}{2}m_{n-1|}q_x$.

Thus the persons altogether live

$$=\frac{m}{2}(q_x+3\,{}_{1|}q_x+\ .\ .\ .\ .\ .\ ),$$

and the average duration will be

$$=\tfrac{1}{2}(q_x+3\,{}_{1|}q_x+5\,{}_{2|}q_x+\ .\ .\ .\ .\ .\ )\,.$$

(40).—Assuming the lives to die upon the average in the middle of the year the mean age at death is equal to

$$\frac{d_x \times (x+\tfrac{1}{2})+d_{x+1}(x+1\tfrac{1}{2})+d_{x+2}(x+2\tfrac{1}{2})+\ldots\ldots}{l_x}$$

$$=\frac{(l_x-l_{x+1})(x+\tfrac{1}{2})+(l_{x+1}-l_{x+2})(x+1\tfrac{1}{2})+(l_{x+2}-l_{x+3})(x+2\tfrac{1}{2})+\ldots\ldots}{l_x}$$

$$=(x+\tfrac{1}{2})+\frac{l_{x+1}+l_{x+2}+l_{x+3}+\ldots\ldots}{l_x}$$

$$=x+\mathring{e}_x.$$

(41).—§ [21].

The curtate expectation of two joint lives (x) and (y) is the integral number of years which lives at those ages will, upon the average, jointly survive according to the given table of mortality, and is obtained by the formula

$$\frac{l_{x+1}l_{y+1}+l_{x+2}l_{y+2}+\ldots\ldots}{l_xl_y},$$

which is obviously equal to the expression given in the question.

Chapter IV.

(42).—§§ [1], [6].

(43).—§ [6]. (*J.I.A.*, xxvii, 158.)

(44).—§§ [3], [8].

The following is an additional formula involving the ordinary approximation for the value of μ_x, but without making any assumption as to the distribution of deaths.

$$Q^1_{xy}=\int {}_tp_{xy}\cdot\mu_{x+t}\cdot dt$$

$$=\int\frac{l_{x+t}\cdot l_{y+t}}{l_{xy}}\mu_{x+t}\cdot dt$$

but $\mu_{x+t}=\dfrac{l_{x+t-1}-l_{x+t+1}}{2l_{x+t}}$ nearly

the above $=\displaystyle\int\frac{l_{y+t}\cdot l_{x+t-1}-l_{y+t}\cdot l_{x+t+1}}{2l_{xy}}dt$

$$=\frac{1}{2}\int\left(\frac{l_{y+t}\cdot l_{x+t-1}}{l_{x-1}l_y}\cdot\frac{l_{x-1}}{l_x}-\frac{l_{y+t}\cdot l_{x+t+1}}{l_{x+1}\cdot l_y}\cdot\frac{l_{x+1}}{l_x}\right)dt$$

$$=\frac{1}{2}\left\{\frac{\mathring{e}_{x-1.y}}{p_{x-1}}-p_x\cdot\mathring{e}_{x+1.y}\right\}.$$

(45).—Let the year be divided into m parts. Then the probability of (x) dying in the first of such parts and of (y) dying afterwards, is

$$=\frac{1}{m}\cdot\frac{m-1}{m}.$$

The probability of (x) dying in the second interval and (y) in a later interval, is

$$=\frac{1}{m}\cdot\frac{m-2}{m}.$$

The probability of (x) dying in the nth interval and (y) in a later interval, is

$$=\frac{1}{m}\cdot\frac{m-n}{m}.$$

The probability of (x) dying in the $(m-1)$th interval and (y) in the mth, is

$$=\frac{1}{m}\cdot\frac{1}{m}.$$

Summing these expressions, we obtain

$$\frac{1}{m}\left(\frac{m-1}{m}+\frac{m-2}{m}+\ldots+\frac{1}{m}\right)=\frac{m-1}{2m}.$$

If m be infinitely large this becomes in the limit $=\frac{1}{2}$.

Or, if t represents the fraction of the year elapsed from its commencement to the death of (y), then, since (x) is equally likely to die in any portion of the year, the probability that he will die in the interval t (*i.e.*, before y) $=t$; since t may have any value from 0 to 1, the total probability that (x) dies before $(y)=\int_0^1 t\,dt=\frac{1}{2}$.

(46).—§ [3]. $Q^1_{xy}=\frac{1}{2}\left(1-\frac{e_{x:y-1}}{p_{y-1}}+\frac{e_{x-1\,y}}{p_{x-1}}\right)$.

Since the expectation of life is equal to the annuity when the rate of interest $=0$, $e_{xy}=\frac{N'_{xy}}{l_{xy}}$ and the above becomes

$$=\frac{1}{2}\left(1-\frac{N'_{x.y-1}}{l_{x.y-1}}\cdot\frac{l_{y-1}}{l_y}+\frac{N'_{x-1.y}}{l_{x-1.y}}\cdot\frac{l_{x-1}}{l_x}\right)$$

$$=\frac{1}{2}\left(1-\frac{N'_{x.y-1}}{l_{xy}}+\frac{N'_{x-1.y}}{l_{xy}}\right)$$

$$=\frac{l_{xy}+N'_{x-1.y}-N'_{x.y-1}}{2l_{xy}}.$$

(47).—§ [13]. If, in the expression

$$1-{}_tp_x(1-Q^{1}_{x+t.y}),$$

the approximation given in the solution of Example (44) be adopted, we have, as an approximate value,

$$Q^{1}_{x:\overline{y(t)}}=1-{}_tp_x+\tfrac{1}{2}({}_{t-1}p_xe_{x+t-1:y}-{}_{t+1}p_xe_{x+t+1:y}).$$

The value of this probability may also be approximately found by substituting another life z, whose expectation of life exceeds that of y by t years; then

$$Q^{1}_{x.\overline{y(t)}}=Q^{1}_{xz} \text{ approximately.}$$

(48).—§ [14].

(49).—§§ [18], [21], [22].

Chapter V.

(50).—§§ [7]-[21].

$$\begin{aligned} L_x &=\tfrac{1}{2}(l_x+l_{x+1}) \\ N'_x &=\Sigma l_{x+1}=l_x\times e_x \\ T_x &=\Sigma L_x=l_x\times \mathring{e}_x \\ Y_x &=\tfrac{1}{2}T_x+\Sigma T_{x+1}. \end{aligned}$$

(51).—Assuming that the age-distribution of the community, that is, the ratio of the number living at any age x to the total population, $=\frac{L_x}{T_0}$, does not vary at different periods for any given value of x, let $L_0, L_1, \ldots$ represent the number of persons living between the ages 0 and 1, 1 and 2, . . .; then if x be the age at which the pension is to commence, the number of persons above that age would be $L_x+L_{x+1}+\&c.$ $=T_x$, and the number of persons below age x will be T_0-T_x: hence (upon the supposition that money does not yield interest) the contributions required from each of the latter to provide a pension of £1 per annum for each of the former$=\dfrac{T_x}{T_0-T_x}$.

.—Let x be the age at which promotions are made from the o the intermediate class, and y that of promotion to the senior

en

$$\begin{aligned} T_{20}-T_x &=\tfrac{3}{6}(T_{20}-T_{60}) \\ T_x-T_y &=\tfrac{2}{6}(T_{20}-T_{60}) \\ T_y-T_{60} &=\tfrac{1}{6}(T_{20}-T_{60}) \end{aligned}$$

The first equation determines the value of x, and either of the others the value of y.

The above solution will only hold good provided the mortality experienced agrees with that of the table used, and that there are no withdrawals otherwise than by death.

(53).—The average age of the population at any time will be

$$\frac{(L_0\times\tfrac{1}{2})+(L_1\times1\tfrac{1}{2})+(L_2\times2\tfrac{1}{2})+\ldots\ldots}{L_0+L_1+L_2+\ldots\ldots}$$

and the total number of years to be lived by

the L_0 persons will be $=\tfrac{1}{2}L_0+L_1+L_2+\ldots\ldots$
the L_1 ,, ,, $=\tfrac{1}{2}L_1+L_2+\ldots\ldots$
the L_2 ,, ,, $=\tfrac{1}{2}L_2+\ldots\ldots$

for the whole population

$$=\tfrac{1}{2}L_0+1\tfrac{1}{2}L_1+2\tfrac{1}{2}L_2+\ldots\ldots$$

and the average expectation

$$=\frac{\tfrac{1}{2}L_0+1\tfrac{1}{2}L_1+2\tfrac{1}{2}L_2+\ldots\ldots}{L_0+L_1+L_2+\ldots\ldots}$$

(54).—Since the number of annual births must equal the number of deaths, $l_0=209$, and

$$\text{The average age at death}=\frac{T_0}{l_0}=\frac{10,000}{209}=47\cdot85.$$

CHAPTER VI.

(55).—Provided the formula employed to represent the number living is such that its sum for successive values of x can readily be represented by an algebraical expression, the values of the various annuity and assurance benefits can be similarly expressed, thus obviating the use of numerical tables. Such an expression is that given by De Moivre,

$$l_x=86-x.$$

Messrs. Gompertz and Makeham have also given formulas

$$l_x=k(g)^{c^x}$$

and $$l_x=ks^x(g)^{c^x}$$

The sum of these expressions for successive values of x can only be expressed in the form of series.

For advantages in the calculation of joint-life annuities, see Chap. vi, § [69], and Chap. xii, §§ [20], [26].

(56).—Let

$l_x, l_{x+1}, l_{x+2}, \ldots\ldots$ be represented by $ar^x, ar^{x+1}, ar^{x+2}, \ldots\ldots$

then
$$q_x = \frac{d_x}{l_x} = \frac{ar^x - ar^{x+1}}{ar^x} = (1-r)$$

$$q_{x+n} = \frac{d_{x+n}}{l_{x+n}} = \frac{ar^{x+n} - ar^{x+n+1}}{ar^{x+n}} = (1-r)$$

$$\mathring{e}_x = \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + l_{x+3} + \ldots\ldots}{l_x}$$

$$= \frac{1}{2} + \frac{ar^{x+1} + ar^{x+2} + ar^{x+3} + \ldots\ldots + ad\ inf.}{ar^x}$$

$$= \frac{1}{2} + r + r^2 + r^3 + \ldots\ldots ad\ inf.$$

$$= \frac{1}{2} + \frac{r}{1-r} = \frac{(1+r)}{2(1-r)} = \frac{2-q}{2q}.$$

(57).—§§ [7], [8], [10].

(58).—§§ [13]-[15].

(59).—
$$\log l_x = \log(ks^x g^{c^x})$$
$$= \log k + x \log s + c^x \log g$$
$$\log l_{x+1} = \log k + (x+1)\log s + c^{x+1}\log g$$
$$\log p_x = \log l_{x+1} - \log l_x$$
$$= \log s + (c^{x+1} - c^x)\log g$$
$$= \log s + c^x(c-1)\log g.$$

To find the value of μ_x, we have

$$\log_\epsilon l_x = \log_\epsilon k + x\log_\epsilon s + c^x \log_\epsilon g$$

$$\mu_x = -\frac{d}{dx}\log_\epsilon l_x$$

$$= -\log_\epsilon s - (\log_\epsilon g \log_\epsilon c)c^x.$$

seen that this expression is of the form

$$\mathrm{A} + \mathrm{B}c^x,$$

B, and c are constants (§§ [14], [15]).

—§§ [19], [20].

CHAPTER VII.

(61).—$ {}_{21}E_{10} = \dfrac{v^{11} l_{21}}{l_{10}} = \cdot 690. $

(62).—§§ [5], [6].

(63).—In § [25] it has been proved that $a_x < a_{\overline{e_x}|}$,

But $$A_x = 1 - d(1 + a_x)$$

while $$v^{(e_x+1)} = 1 - d(1 + a_{\overline{e_x}|})$$

$$\therefore \quad A_x > v^{(e_x+1)}$$

multiplying both sides by $(1+i)^{\frac{1}{2}}$,

$$(1+i)^{\frac{1}{2}} A_x > v^{(e_x+\frac{1}{2})}$$

or $$\overline{A}_x > v^{(\mathring{e}_x)}$$

that is, the value of an assurance payable at the instant of the death of (x) is greater than that of a sum certain payable at the expiration of the complete expectation.

(64).—§§ [30], [39]-[44].

(65).—From the investment of a unit in connection with a life (x) we can obtain an annuity of i per annum for $(n-1)$ years, and an assurance of $(1+i)$ payable upon the attainment of age $(x+n)$ or at previous death: that is,

$$1 = i\,{}_{|n-1}a_x + (1+i)\,.\,A_{\overline{xn}|},$$

or
$$\begin{aligned} 1 &= i\,{}_{|n-1}a_x + (1+i)({}_{|n}A_x + {}_{n}E_x) \\ &= i\,{}_{|n-1}a_x + (1+i)({}_{|n}A_x + {}_{|n}a_x - {}_{|n-1}a_x) \\ &= (1+i)({}_{|n}A_x + {}_{|n}a_x) - {}_{n-1}a_x \end{aligned}$$

whence $${}_{|n}A_x = v(1 + {}_{|n-1}a_x) - {}_{|n}a_x.$$

(66).—(*a*) $v(1 + {}_{|n-1}a_x)$ represents the value of an annuity of 1 for n years payable at the end of each year, provided the life (x) be in existence at the beginning of the year.

${}_{|n}a_x$ represents the value of a similar annuity, provided the life (x) be in existence at the end of the year.

The difference between the value of these two annuities represents the present value of 1 payable at the end of the year in which the life (x) fails, provided that event happen within the n years.

(β) If an assurance on (x) were payable at the end of the first year, its value would be v; but, under the conditions of the given formula, the sum is payable after n years or at the end of the year of previous death: we must therefore deduct from v the value of the interest thereon for the term of $(n-1)$ years during the life of (x)—that is, $iv|_{n-1}a_x$, or $d|_{n-1}a_x$;

$$\therefore \quad A_{x\overline{n}|} = v - d|_{n-1}a_x.$$

(67).—A capital of 1 will provide a payment of i at the end of each of the $(t-1)$ years while the life (x) is living, with a return of the capital and one year's interest at the end of the period or at the end of the year in which the life fails: thus,

$$1 = i|_{t-1}a_x + (1+i)A_{x\overline{t}|},$$

whence
$$A_{x\overline{t}|} = \frac{1 - i_{t-1}a_x}{1+i}$$

$$\frac{1-i|_{t-1}a_x}{1+i} = v-(1-v)|_{t-1}a_x$$

$$= v-(1-v)\frac{N_x - N_{x+t-1}}{D_x}$$

$$= \frac{vD_x - (1-v)N_x + (1-v)N_{x+t-1}}{D_x}$$

$$= \frac{vN_{x-1} - N_x - vN_{x+t-1} + (N_{x+t} + D_{x+t})}{D_x}$$

$$= \frac{M_x - M_{x+t} + D_{x+t}}{D_x}.$$

(68).—§§ [53], [54].

It should be noted that in Dr. Farr's tables

$$N_{xy} = D_{xy} + D_{x+1:y+1} + \dots\dots$$

(69).— (α) $\dfrac{M_x - M_{x+m+n} + D_{x+m+n}}{N_{x-1} - N_{x+m-1}}$

(β) $\dfrac{M_x - M_{x+n}}{D_x}$

(γ) (i) $\dfrac{M_{x+n}}{N_{x-1} - N_{x+n-1}}$

(ii) $\dfrac{M_{x+n}}{N_{x-1}}$

(70).—§ [82].

The annuity is equal to

$$\Sigma_1^{(\omega-x)} v^t \{1-(1-{}_tp_x)^n\} = v^t \Big\{ n\,{}_tp_x - \frac{n(n-1)}{\underline{|2}}({}_tp_x)^2 + \frac{n(n-1)(n-2)}{\underline{|3}}({}_tp_x)^3 - \ldots\ldots \pm ({}_tp_x)^n \Big\}$$

$$= \Sigma_1^{(\omega-x)} v^t \Big\{ n\,{}_tp_x - \frac{n(n-1)}{\underline{|2}}\,{}_tp_{xx} + \frac{n(n-1)(n-2)}{\underline{|3}}\,{}_tp_{xxx} - \ldots \pm {}_tp_{xxx\ldots(n)} \Big\}$$

$$= na_x - \frac{n(n-1)}{\underline{|2}}a_{xx} + \frac{n(n-1)(n-2)}{\underline{|3}}a_{xxx} - \ldots\ldots \pm a_{xxx\ldots\ldots(n)}$$

(71).—§ [85].

(72).—$a_{xyz}^{\lvert 2\rvert} = \Sigma v^t[{}_tp_{xy}(1-{}_tp_z) + {}_tp_{xz}(1-{}_tp_y) + {}_tp_{yz}(1-{}_tp_x) + {}_tp_{xyz}]$

$= \Sigma v^t[{}_tp_{xy} + {}_tp_{xz} + {}_tp_{yz} - 2{}_tp_{xyz}]$

$= a_{xy} + a_{xz} + a_{yz} - 2a_{xyz}.$

If an annuity of 1 be purchased on the joint existence of each possible pair of lives, we shall evidently have a total annuity payment of 3 so long as all the lives are in existence, and as each of the lives is included in two pairs, the failure of any life will reduce the annuity to 1, and the failure of a second life will extinguish it. It is evident, therefore, that the required case will be exactly met by the purchase of an annuity of 1 on each possible pair of lives, *less* an annuity of 2 during the joint existence of all three lives—that is,

$$(a_{xy} + a_{xz} + a_{yz} - 2a_{xyz}).$$

(73).—§ [93]-[97]. (*J.I.A.*, xxiii, 244.)

(74).—§ [99].—The value of this annuity is approximately equal to a_{xz} where the value of z is deduced from the formula $e_z = e_y + t$.

(75).—§ [103].

(76).—§§ [105]-[107].

(77).—§ [113].

(78).— (α) $\dfrac{D_{x+n}}{D_x}$

(β) $\dfrac{D_{x+n:y+n}}{D_{xy}}$

(γ) $\dfrac{D_{x+n}}{D_x} + \dfrac{D_{y+n}}{D_y} - \dfrac{D_{x+n:y+n}}{D_{xy}}$

(δ) $\dfrac{D_{x+n}}{D_x} - \dfrac{D_{x+n:y+n}}{D_{xy}}$

(79).— $$(\alpha)\quad a_{xy}+\frac{m}{n}(a_x-a_{xy})+\frac{m}{n}(a_y-a_{xy})$$
$$=\frac{m}{n}(a_x+a_y)+\left(1-\frac{2m}{n}\right)a_{xy}$$

$$(\beta)\quad a_{xy}+\frac{m}{n}(a_y-a_{xy})+(a_x-a_{xy})$$
$$=a_x+\frac{m}{n}(a_y-a_{xy}).$$

(80).—$A_{\overline{xxx}}=1-d(1+a_{\overline{xxx}})$
$=1-d(1+3a_x-3a_{xx}+a_{xxx})$
$=[1-d(1+a_{xxx})]-3[1-d(1+a_{xx})]+3[1-d(1+a_x)]$
$=A_{xxx}-3(A_{xx}-A_x)$.

The annual premiums would not follow the same rule, for

$$P_{\overline{xxx}}=\frac{A_{\overline{xxx}}}{1+a_{\overline{xxx}}}=\frac{A_{xxx}-3(A_{xx}-A_x)}{(1+a_{xxx})-3(a_{xx}-a_x)}$$

while $$P_{xxx}-3(P_{xx}-P_x)=\frac{A_{xxx}}{1+a_{xxx}}-\frac{3A_{xx}}{1+a_{xx}}+\frac{3A_x}{1+a_x}.$$

(81).— $${}_{m|}a_x+{}_{n|}a_{x|y}={}_{m|}a_x+{}_{n|}a_y-{}_{n|}a_{xy}$$
$$={}_{m|}a_x-{}_{n|}a_x+{}_{n|}a_{\overline{xy}}$$
$$={}_{n|m-n}a_x+{}_{n|}a_{\overline{xy}}.$$

(82).—Let the several ages be w, x, y, z, then the interest of (w) is equal to

$$\tfrac{1}{4}a_{w.\overset{[3]}{\overline{xyz}}}+\tfrac{1}{3}a_{w.\overset{[2]}{\overline{xyz}}}+\tfrac{1}{2}a_{w.\overset{[1]}{\overline{xyz}}}+a_{w.\overset{[0]}{\overline{xyz}}}$$

which may be symbolically expressed (§ [85])

$$\tfrac{1}{4}a_w(Z^3)+\tfrac{1}{3}a_w(Z^2-3Z^3)+\tfrac{1}{2}a_w(Z^1-2Z^2+3Z^3)+a_w(Z^0-Z^1+Z^2-Z^3)$$
$$=a_w(Z^0-\tfrac{1}{2}Z^1+\tfrac{1}{3}Z^2-\tfrac{1}{4}Z^3)$$
$$=a_w-\tfrac{1}{2}(a_{wx}+a_{wy}+a_{wz})+\tfrac{1}{3}(a_{wxy}+a_{wyz}+a_{wxz})-\tfrac{1}{4}a_{wxyz}.$$

For two lives (w), (x), we have

$$\tfrac{1}{2}a_{wx}+(a_w-a_{wx})=a_w-\tfrac{1}{2}a_{wx}=a_w(Z^0-\tfrac{1}{2}Z^1).$$

lives (w), (x), (y),

$$\tfrac{1}{3}a_{wxy}+\tfrac{1}{2}(a_{wx}+a_{wy}-2a_{wxy})+(a_w-a_{wx}-a_{wy}+a_{wxy})$$
$$=a_w-\tfrac{1}{2}(a_{wx}+a_{wy})+\tfrac{1}{3}a_{wxy}$$
$$=a_w(Z^0-\tfrac{1}{2}Z^1+\tfrac{1}{3}Z^2).$$

For four lives (as above),

$$a_w(Z^0-\tfrac{1}{2}Z^1+\tfrac{1}{3}Z^2-\tfrac{1}{4}Z^3).$$

The expression for n lives is thus evidently

$$=a_w\left(Z^0-\tfrac{1}{2}Z^1+\tfrac{1}{3}Z^2-\tfrac{1}{4}Z^3+\ldots\ldots\pm\frac{1}{n}Z^{n-1}\right)$$

which may be symbolically expressed as

$$a_w\left[\frac{\log_\epsilon(1+Z)}{Z}\right]$$

and is equal to

$$a_w-\tfrac{1}{2}(a_{wx}+a_{wy}+a_{wz}+\ldots\ldots)+\tfrac{1}{3}(a_{wxy}+a_{wxz}+a_{wyz}+\ldots\ldots)$$
$$-\tfrac{1}{4}(a_{wxyz}+\ldots\ldots)+\ldots\ldots\pm\frac{1}{n}(a_{wxyz\ldots\ldots(n)})$$

If all the lives are of the same age (x), the expression becomes

$$a_x-\frac{n-1}{\underline{|2}}a_{xx}+\frac{(n-1)(n-2)}{\underline{|3}}a_{xxx}-\frac{(n-1)(n-2)(n-3)}{\underline{|4}}a_{xxxx}+\ldots$$
$$\pm\frac{1}{n}a_{xxxx\ldots\ldots(n)}.$$

CHAPTER VIII.

(83).—A capital of 1 will produce an income of i per annum during the continuance of any status, the value of which is by hypothesis iQ; and, at the end of the year in which the status fails, a payment of $(1+i)$, the present value of which is $P(1+i)$, hence

$$1=iQ+P(1+i),$$

and

$$P=\frac{1-iQ}{1+i}.$$

If the status be a single life (x), we have

$$1=ia_x+A_x(1+i)$$

whence (multiplying by D_x)

$$D_x=iN_x+(1+i)M_x.$$

(84).—§ [33].

(85).— $$A_{x\overline{n}|}=1-d(1+{}_{|n-1}a_x)$$

$$P_{x\overline{n}|}=\frac{1}{1+{}_{|n-1}a_x}-d.$$

Hence, applying the principle set out in § [33], the tables must be entered with the value of a temporary annuity on the life (x) for ($n-1$) years.

(86).—§ [32]. $$P_x=\frac{A_x}{1+a_x}=\frac{dA_x}{1-A_x};$$

hence, by calculating the value of this expression for successive values of A_x, we have the table required.

(87).— $$\frac{P_x}{A_x}=\frac{1}{1+a_x}=P_x+d$$

$$\therefore\quad d=\frac{P_x}{A_x}-P_x$$

$$d=P_x\left(\frac{1-A_x}{A_x}\right)$$

and $$i=\frac{d}{1-d}=\frac{P_x(1-A_x)}{A_x-P_x(1-A_x)}.$$

(88).— $$1-ia_x=(1+i)A_x$$

$$i=\frac{1-A_x}{a_x+A_x}=\frac{\cdot697342}{17\cdot433558}=\cdot04.$$

(89).—§ [24]. (Chap. xviii, § [52].)

CHAPTER IX.

(90).—§§ [6]-[9].

(91).—It is shown in §§ [21], [22], that the value of an annuity upon a life (x), payable m times a year,

$$a_x^{(m)}=a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu_x+\delta)$$

$$\therefore\quad a_x^{(2)}=a_x+\tfrac{1}{4}-\tfrac{1}{16}(\mu_x+\delta)$$

$$a^{(4)}=a_x+\tfrac{3}{8}-\tfrac{5}{64}(\mu_x+\delta);$$

the amount of the error is therefore $+\frac{1}{16}(\mu_x+\delta)$ in the case of a half-yearly annuity, and $+\frac{5}{64}(\mu_x+\delta)$ in the case of a quarterly annuity.

(92).—§ [38]. $${}_{\frac{1}{2m}|}\mathbf{a}_x^{(m)} = \text{approximately } \tfrac{1}{2}(a_x^{(m)}+\mathbf{a}_x^{(m)})$$

$$=\frac{1}{2}\left(a_x+\frac{m-1}{2m}+a_x+\frac{m+1}{2m}\right)$$

$$=(a_x+\tfrac{1}{2}) \text{ approximately.}$$

(93).—§§ [39]–[41].

(94).—From the result in § [41], we have

$$P_x^{(2)}=\frac{P_x}{1-\frac{1}{4}(P_x+d)} \quad \text{approximately,}$$

and the half-yearly premium

$$=\frac{P_x^{(2)}}{2}=\frac{\frac{1}{2}P_x}{1-\frac{1}{4}(P_x+d)} \quad \text{approximately.}$$

(95).—$a_{wxyz\ldots}^{(m)}=a_{wxyz\ldots}+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu_w+\mu_x+\mu_y+\mu_z+\ldots+\delta)$.

If $m=\infty$, this becomes

$$\bar{a}_{wxyz\ldots}=a_{wxyz\ldots}+\tfrac{1}{2}-\tfrac{1}{12}(\mu_w+\mu_x+\mu_y+\mu_z+\ldots\ldots+\delta)$$

$$\bar{a}_{y|x}=\bar{a}_x-\bar{a}_{xy}$$

$$=(a_x-a_{xy})+\frac{\mu_y}{12}.$$

(96).— $$\frac{v^n{}_np_xa_{x+n}^{(2)}}{1+|_na_x}=\frac{v^n{}_np_x(a_{x+n}+\frac{1}{4})}{1+|_na_x} \quad \text{approximately.}$$

$$=\frac{\dfrac{D_{x+n}}{D_x}\cdot\dfrac{N_{x+n}+\frac{1}{4}D_{x+n}}{D_{x+n}}}{\dfrac{N_{x-1}-N_{x+n}}{D_x}}$$

$$=\frac{N_{x+n}+\frac{1}{4}D_{x+n}}{N_{x-1}-N_{x+n}}.$$

This expression may also be written $\dfrac{N_{x+n-\frac{1}{4}}}{N_{x-1}-N_{x+n}}$.

(97).— $${}_{|n}a_x^{(4)} = a_x^{(4)} - v^n{}_np_x a_{x+n}^{(4)}$$
$$= (a_x + \tfrac{3}{8}) - v^n{}_np_x(a_{x+n} + \tfrac{3}{8})$$
$$= \frac{N_x}{D_x} - \frac{D_{x+n}}{D_x}\cdot\frac{N_{x+n}}{D_{x+n}} + \frac{3}{8}\left(1 - \frac{D_{x+n}}{D_x}\right)$$
$$= \frac{N_x - N_{x+n} - \frac{3}{8}D_{x+n}}{D_x} + \frac{3}{8}.$$

This expression may also be written $\dfrac{N_{x-\frac{3}{8}} - N_{x+n-\frac{3}{8}}}{D_x}$.

Chapter X.

(98).—§§ [12], [16], [17].

(99).—Let i be the *effective* rate of interest, and let

$$j_{(2)} = 2[(1+i)^{\frac{1}{2}} - 1]$$
$$j_{(m)} = m[(1+i)^{\frac{1}{m}} - 1]$$

be the *nominal* rates when interest is payable half-yearly, or m times a year respectively.

Then we have

$$1 = j_{(2)}a_x^{(2)} + \left(1 + \frac{j_{(2)}}{2}\right)A_x^{(2)}$$
$$1 = j_{(m)}a_x^{(m)} + \left(1 + \frac{j_{(m)}}{m}\right)A_x^{(m)};$$

whence (1) $$A_x^{(2)} = \frac{1 - j_{(2)}a_x^{(2)}}{1 + \frac{j_{(2)}}{2}};$$

(2) $$A_x^{(m)} = \frac{1 - j_{(m)}a_x^{(m)}}{1 + \frac{j_{(m)}}{m}}.$$

$j_{(m)} = \delta$, and $\dfrac{j_{(m)}}{m} = 0$, and we have

(3). $$\bar{A}_x = 1 - \delta\bar{a}_x.$$

Chap. ix, § [13]; Chap. x, § [7].

Let the year be divided into m equal intervals; then, as it is [illegible] that a given life (x) will fail during any of these intervals, the present value, at the commencement of the year of death, of the sum payable at the instant of death will be, upon the average,

$$\frac{1}{m}\left(v^{\frac{1}{m}}+v^{\frac{2}{m}}+\ldots\ldots+v^{\frac{m-1}{m}}+v^{\frac{m}{m}}\right)=\frac{1-v}{m[(1+i)^{\frac{1}{m}}-1]}.$$

Let $m=\infty$, then this expression becomes $\frac{1-v}{\delta}=\frac{i}{\delta}(v)$.

If the sum be payable at the end of the year of death, the present value at the commencement of that year will be uniformly v. Thus

$$\overline{A}_x : A_x = \frac{i}{\delta}(v) : v;$$

or $$\overline{A}_x=\frac{i}{\delta}(A_x)=\left(1+\frac{i}{2}-\frac{i^2}{12}+\frac{i^3}{24}\right)A_x \text{ approximately.}$$

(101).— $$1=i|_{n-1}a_x+(1+i)A_{x\overline{n}|}$$

$$1=\delta|_n\bar{a}_x+\overline{A}_{x\overline{n}|},$$

whence $$\overline{A}_{x\overline{n}|}=1-\delta|_n\bar{a}_x.$$

(102).—§ [20].

CHAPTER XI.

(103).—Chap. ix, § [22].

$$a_x^{(m)}=a_x+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu_x+\delta)$$

If $m=\infty$,

$$\bar{a}_x=a_x+\tfrac{1}{2}-\tfrac{1}{12}(\mu_x+\delta),$$

whence, eliminating the element of interest,

$$\bar{e}_x \text{ or } \mathring{e}_x=e_x+\frac{1}{2}-\frac{\mu_x}{12}.$$

(104).—§§ [3], [5], [7].

(105).— $$\mathring{a}_x^{(2)}=a_x^{(2)}+\frac{\overline{A}_x}{4}-\frac{\mu_x}{48}, \text{ formula (11),}$$

$$=\left(a_x+\frac{1}{4}-\frac{\mu_x+\delta}{16}\right)+\frac{\overline{A}_x}{4}-\frac{\mu_x}{48}$$

$$=a_x+\left(\frac{1+\overline{A}_x}{4}-\frac{\mu_x}{12}-\frac{\delta}{16}\right).$$

(106).— $${}_{t|}\mathring{a}^{(4)}_{xyz} = \frac{D_{x+t:y+t:z+t}}{D_{xyz}}\,\mathring{a}^{(4)}_{x+t:y+t:z+t}$$

$$= \frac{D_{x+t:y+t:z+t}}{D_{xyz}}\left(a^{(4)}_{x+t:y+t:z+t} + \frac{\bar{A}_{x+t:y+t:z+t}}{8} - \frac{\mu_{x+t:y+t:z+t}}{192}\right)$$

$$= {}_{t|}a^{(4)}_{xyz} + \frac{{}_{t|}\bar{A}_{xyz}}{8} - \frac{{}_{t|}\mu_{xyz}}{192}.$$

(107).—§ [11].

CHAPTER XII.

(108).—§§ [3]-[11].
(109).—§ [12].
(110).—§§ [17]-[22].

$${}_{n}p_{xy} = g^{c^x(c^n-1)} \times g^{c^y(c^n-1)}$$

$$= g^{(c^x+c^y)(c^n-1)}$$

but $${}_{n}p_{w} = g^{c^w(c^n-1)};$$

therefore if w is deduced from the equation

$$c^x + c^y = c^w,$$

or from the equivalent relation

$$\mu_x + \mu_y = \mu_w,$$

the probabilities for the single life (w) will be identical for all values of n, with those for the joint lives (x), (y).

(111).—Chap. vi, § [28]; Chap. xii, §§ [26], [27].
(112).—§§ [43]-[45]. (*J.I.A.*, xv, 401.)
(113).—§§ [46]-[51].

CHAPTER XIII.

(114).—§§ [3]–[5].—The single premium may also be approximately obtained by the following process, in which no assumption is made as to the distribution of deaths. [See also Solution of Ex. 44, p. 64.]

$$\bar{A}^1_{xy}=\int v^t\frac{l_{x+t}\cdot l_{y+t}}{l_x l_y}\cdot\mu_{x+t}dt$$

$$=\int v^t\frac{l_{y+t}}{l_y}\cdot\frac{l_{x+t-1}-l_{x+t+1}}{2l_x}\,dt,\text{ approximately}$$

$$=\frac{1}{2}\left(\frac{\bar{a}_{y:x-1}}{p_{x-1}}-p_x\bar{a}_{y:x+1}\right)\text{ approximately.}\quad(\textit{J.I.A.},\text{ xv, 123.})$$

(115).— $$A^2_{xy}=A_x-A^1_{xy}$$

The life y must be medically examined, but if y is much older than x, both lives should be examined.

(116).— $${}_{|n}A^1_{xy}=A^1_{xy}-{}_{n|}A^1_{xy}$$

$$=A^1_{xy}-v^n{}_np_{xy}A^{\ 1}_{x+n:y+n}$$

$$=\frac{1}{2}\left({}_{|n}A_{xy}+\frac{{}_{|n}a_{x-1:y}}{p_{x-1}}-\frac{{}_{|n}a_{x:y-1}}{p_{y-1}}\right)$$

(117).—§§ [14], [15].

(118).— $${}_{|t}A^1_{xy}=\frac{M^1_{xy}-M^{\ 1}_{x+t\cdot y+t}}{D_{xy}}$$

$${}_{|t}A_{xy}=\frac{v(N_{x-1:y-1}-N_{x+t-1:y+t-1})-(N_{x:y}-N_{x+t:y+t})}{D_{xy}}$$

$${}_{|t}A_x=\frac{v(N_{x-1}-N_{x+t-1})-(N_x-N_{x+t})}{D_x}$$

$${}_{|t}A_y=\frac{v(N_{y-1}-N_{y+t-1})-(N_y-N_{y+t})}{D_y}$$

then $${}_{|t}A_{x\overset{1}{y}}={}_{|t}A_{xy}-{}_{|t}A^1_{xy}$$

$${}_{|t}A^2_{xy}={}_{|t}A_x-{}_{|t}A^1_{xy}$$

$${}_{|t}A_{x\overset{2}{y}}={}_{|t}A_{\overline{xy}}-{}_{|t}A^2_{xy}$$

$$={}_{|t}A_y-{}_{|t}A_{x\overset{1}{y}}.$$

(119).—§§ [21]-[23].

An approximate value for the single premium can also be found as follows [see Solution to Ex. (114)]:

$$\overline{\mathrm{A}}^1_{x.y\overline{(t|)}}={}_{|t}\overline{\mathrm{A}}_x+\int v^{t+n}\frac{1}{l_xl_y}\,l_{x+t+n}\cdot l_{y+n}\cdot\mu_{x+t+n}\cdot dn$$

$$={}_{|t}\overline{\mathrm{A}}_x+\int v^{t+n}\frac{1}{l_xl_y}\,l_{y+n}\cdot\frac{l_{x+t+n-1}-l_{x+t+n+1}}{2}\cdot dn \text{ approximately}$$

$$={}_{|t}\overline{\mathrm{A}}_x+\frac{v^t}{2}\left({}_{t-1}p_x\cdot\bar{a}_{y:x+t-1}-{}_{t+1}p_x\cdot\bar{a}_{y\ x+t+1}\right)$$

$$={}_{|t}\overline{\mathrm{A}}_x+\frac{\mathrm{D}_{x+t}}{2\mathrm{D}_x}\left(\frac{\bar{a}_{y:x+t-1}}{p_{x+t-1}}-p_{x+t}\cdot\bar{a}_{y:x+t+1}\right)$$

If $t=1$ this becomes

$$=v^{\frac{1}{2}}q_x+\frac{\mathrm{D}_{x+1}}{2\mathrm{D}_x}\left(\frac{\bar{a}_{y:x}}{p_x}-p_{x+1}\cdot\bar{a}_{y:x+2}\right):$$

The value of this assurance may also be found approximately by substituting another life z, whose expectation of life exceeds that of y by t years, then

$$\mathrm{A}^1_{x:y\overline{(t|)}}=\mathrm{A}^1_x \quad \text{approximately.}$$

(120).—We have

$$\mathrm{A}^1_{48:\overline{75:70}\overline{(1|)}}\doteq\mathrm{A}^1_{48:\overline{75(1|)}}+\mathrm{A}^1_{48\cdot\overline{70(1|)}}-\mathrm{A}^1_{48\ \overline{75:70(1|)}}.$$

If $a_{75:70}=a_w$, this becomes approximately

$$=\mathrm{A}^1_{48:\overline{75(1|)}}+\mathrm{A}^1_{48:\overline{70(1|)}}-\mathrm{A}^1_{48:\overline{w(1|)}}.$$

Then, if $e_x=e_{75}+1$, $e_y=e_{70}+1$, $e_z=e_w+1$ the expression becomes approximately

$$\doteq\mathrm{A}^1_{48:x}+\mathrm{A}^1_{48:y}-\mathrm{A}^1_{48:z}.$$

Or we may proceed as in the previous example (119), where it is shown that

$$\overline{\mathrm{A}}^1_{x:y(1|)}=v^{\frac{1}{2}}q_x+\frac{\mathrm{D}_{x+1}}{2\mathrm{D}_x}\left(\frac{\bar{a}_{y:x}}{p_x}-p_{x+1}\cdot\bar{a}_{y:x+2}\right) \text{ approximately.}$$

resent case $y=\overline{75.70}$ and $x=48$; the above expression there-omes

$$\overline{\mathrm{A}}^1_{48:\overline{75:70}(1|)}=v^{\frac{1}{2}}q_{48}+\frac{\mathrm{D}_{49}}{2\mathrm{D}_{48}}\left(\frac{\bar{a}_{\overline{70:75}:48}}{p_{48}}-p_{49}\bar{a}_{\overline{70:75}:50}\right),$$

which becomes, if $\bar{a}_{70:75}=\bar{a}_{w}$,

$$=v^{\frac{1}{2}}q_{48}+\frac{D_{49}}{2D_{48}}\left[\frac{\bar{a}_{70:48}+\bar{a}_{75:48}-\bar{a}_{w:48}}{p_{48}}-p_{49}(\bar{a}_{70:50}+\bar{a}_{75:50}-\bar{a}_{w:50})\right].$$

A more accurate solution may be obtained by the use of one of the formulas of approximate summation given in Chap. xxiv. [See Example (265)].

The divisor for the annual premium will be either

$$(1+a_{48:\overline{70:75}})=1+a_{48:70}+a_{48:75}-a_{48:70:75},$$

in the case where it is payable until the failure of the joint existence of (48) and the survivor of (70) and (75); or

$$(1+a_{48:\overline{\overline{70:75}(1|)}})=(1+a_{48})-\frac{D_{49}}{D_{48}}(a_{49}-a_{49:75}-a_{49:70}+a_{49:70:75}),$$

in the case where it is payable until the failure of the joint existence of (48) and the survivor of (70) and (75) and for one year longer if (48) live so long.

(121).—The single premium for the benefit required is equal to

$${}_{n}A^{1}_{xy}+\frac{D_{x+n:y+n}}{D_{xy}}$$

$$=A^{1}_{xy}+\frac{D_{x+n:y+n}}{D_{xy}}(1-A^{1}_{x+n:y+n})$$

$$=\frac{1}{2}\left\{A_{xy}+\frac{a_{x-1:y}}{p_{x-1}}-\frac{a_{x:y-1}}{p_{y-1}}\right\}$$

$$+\frac{D_{x+n:y+n}}{D_{xy}}\left[1-\frac{1}{2}\left\{A_{x+n:y+n}+\frac{a_{x+n-1:y+n}}{p_{x+n-1}}-\frac{a_{x+n:y+n-1}}{p_{y+n-1}}\right\}\right]$$

$$=\frac{1}{2}\left\{1-d(1+a_{xy})+\frac{a_{x-1:y}}{p_{x-1}}-\frac{a_{x:y-1}}{p_{y-1}}\right\}$$

$$+\frac{D_{x+n}}{D_{x}}\cdot\frac{l_{y+n}}{l_{y}}\left[1-\frac{1}{2}\left\{1-d(1+a_{x+n:y+n})+\frac{a_{x+n-1:y+n}}{p_{x+n-1}}-\frac{a_{x+n:y+n-1}}{p_{y+n-1}}\right\}\right],$$

all of which values will be found in the Institute Tables. The annual premium would be obtained by dividing the above expression by

$$(1+{}_{|n-1}a_{xy})=(1+a_{xy})-\frac{D_{x+n}\cdot l_{y+n}}{D_{x}\cdot l_{y}}(1+a_{x+n:y+n}).$$

(122).—§ [25]. The divisor for the annual premium would be $=(1+a_{yz})$.

(123).— $$A^{1.3}_{x:yz}=A_x-A^{2}_{x:yz}$$
$$=A_x-A^1_{xy}-A^1_{xz}+2A^1_{xyz}.$$

(124).—§§ [32], [33]. (*J.I.A.*, xv, 119.)

CHAPTER XIV.

(125).—§ [17]. $a_y-a_{xy}=$

$$\frac{D_{y+1}+D_{y+2}+\ldots}{D_y}-\frac{D_{x+1:y+1}+D_{x+2:y+2}+\ldots}{D_{xy}}$$
$$=\frac{(l_xD_{y+1}+l_xD_{y+2}+\ldots)-(l_{x+1}D_{y+1}+l_{x+2}D_{y+2}+\ldots)}{l_xD_y}$$
$$=\frac{d_xD_{y+1}+(d_x+d_{x+1})D_{y+2}+\ldots}{l_xD_y}$$
$$=\frac{d_xN_y+d_{x+1}N_{y+1}+\ldots}{l_xD_y}=\Sigma\frac{d_{x+t-1}N_{y+t-1}}{l_xD_y}.$$

(126).—§ [11]. $$a_{\overline{t}|}-{}_{|t}a_x.$$

(127).— $$a_{x|\overline{yz}}=a_{\overline{yz}}-a_{x:\overline{yz}}$$
$$=a_y+a_z-a_{yz}-a_{xy}-a_{xz}+a_{xyz}.$$

The divisor for the annual premium is $=(1+a_{x:\overline{yz}})$
$$=1+a_{xy}+a_{xz}-a_{xyz}.$$

(128).—B's interest is equal to
$$\tfrac{1}{2}a_{x|yz}+a_{\overline{xz}|y}$$
$$=\tfrac{1}{2}(a_{yz}-a_{xyz})+(a_y-a_{xy}-a_{yz}+a_{xyz})$$
$$=(a_y-a_{xy})-\tfrac{1}{2}(a_{yz}-a_{xyz})$$
$$=a_{x|y}-\tfrac{1}{2}a_{x|yz}.$$

(129).— $$S=A\,.\,a_{xyz}+\tfrac{4}{5}A(a_{x|yz}+a_{y|xz}+a_{z|xy})$$
$$+\tfrac{2}{5}A(a_{\overline{yz}|x}+a_{\overline{xy}|z}+a_{\overline{xz}|y})$$
$$=A\cdot\frac{2(a_x+a_y+a_z)-a_{xyz}}{5}$$
$$A=\frac{5S}{2(a_x+a_y+a_z)-a_{xyz}}.$$

(130).—The value for the first n years is equal to

$${}_{|n}a_x + {}_{n}a_y - 2\,{}_{|n}a_{xy},$$

and after n years is equal to

$${}_{n|}a_x + {}_{n|}a_y - {}_{n|}a_{xy}$$

The total value is thus

$$a_x + a_y - a_{xy} - {}_{|n}a_{xy} = a_{\overline{xy}} - {}_{|n}a_{xy}.$$

The annual premium would be obtained by dividing the above expression by $(1 + {}_{|n-1}a_{xy})$.

(131).—Single Premium$= {}_{|n}a_x - {}_{|n}a_{xy}$;

Divisor for Annual Premium$= (1 + {}_{|n-1}a_{xy})$.

(132).— $$\begin{aligned} a_{y|\overline{x\overline{t|}}} &= a_{\overline{x\overline{t|}}} - a_{\overline{x\overline{t|}}:y} \\ &= a_x + a_{\overline{t|}} - {}_{|t}a_x - a_{xy} - {}_{|t}a_y + {}_{|t}a_{xy} \\ &= {}_{t|}a_x - {}_{t|}a_{xy} + a_{\overline{t|}} - {}_{|t}a_y \\ &= {}_{t|}a_{y|x} + a_{y|\overline{t|}}. \end{aligned}$$

(133).—For the first t years reckoning from the end of the year of death of (x), the value of the annuity is evidently equal to

$$A_x(1 + a_{\overline{t-1|}})$$

After t years the value is equal to

$$\begin{aligned} &\Sigma_{t+1}^{\infty} v^n \left(\frac{l_{y+n}}{l_y} \cdot \frac{l_x - l_{x+n-t+1}}{l_x} \right) \\ &= \Sigma_{t+1}^{\infty} v^n \left(\frac{l_{y+n}}{l_y} - \frac{l_{y+n} l_{x+n-t+1}}{l_x l_y} \right) \\ &= v^t {}_t p_y (a_{y+t} - a_{x:y+t}) = \frac{D_{y+t}}{D_y} (a_{y+t} - a_{x:y+t}). \end{aligned}$$

The total value is thus equal to

$$A_x(1 + a_{\overline{t-1|}}) + \frac{D_{y+t}}{D_y} (a_{y+t} - a_{x:y+t}).$$

Another solution may be obtained as follows:—The annuity for the first t years is payable if (x) be dead, irrespective of the life (y), and its value is equal to

$$\Sigma_1^t v^n \frac{l_x - l_{x+n}}{l_x} = a_{\overline{t|}} - {}_{|t}a_x$$

After t years, the annuity is payable (i) in the event of (x) having died not more than t years previously, irrespective of the life (y); or (ii) in the event of (x) having died more than t years previously, provided (y) be living: its value is therefore equal to

$$\Sigma_{t+1}^{\infty} v^n\left(\frac{l_{x+n-t}-l_{x+n}}{l_x}+\frac{l_x-l_{x+n-t}}{l_x}\cdot\frac{l_{y+n}}{l_y}\right)$$

$$=v^t(a_x-{}_tp_x . a_{x+t}+{}_tp_y . a_{y+t}-{}_tp_y . a_{x:y+t})$$

$$=v^t a_x-{}_{t|}a_x+{}_{t|}a_y-\frac{D_{y+t}}{D_y}a_{x:y+t}.$$

Adding to this the value for the first t years, we have

$$a_{\overline{t}|}-(1-v^t)a_x+\frac{D_{y+t}}{D_y}(a_{y+t}-a_{x:y+t})$$

This is equal to the value previously deduced, for

$$a_{\overline{t}|}-(1-v^t)a_x=(1-v^t)\left(\frac{1}{i}-\ddot{a}_x\right)=\frac{1-v^t}{d}A_x=A_x(1+a_{\overline{t-1}|}).$$

(134).—The required single premium is equal to

$$a_{y|x}-\frac{D_{x+t.y+t}}{D_{xy}}(a_{y+t|x+t})$$

$$=(a_x-a_{xy})-\frac{D_{x+t.y+t}}{D_{xy}}(a_{x+t}-a_{x+t:y+t}).$$

The divisor for the annual premium is equal to $(1+{}_{\overline{t-1}|}a_{xy})$.

(135).—§ [18].

(136).—§§ [19], [37].

(137).—

(a) $$\frac{A_y-A_{xy}}{a_x-a_{xy}}=\frac{1-d\mathbf{a}_y-1+d\mathbf{a}_{xy}}{a_x-a_{xy}}=\frac{d(a_{xy}-a_y)}{a_x-a_{xy}}$$

$$=\frac{d(a_x-a_{xy})}{a_x-a_{xy}}=d=\frac{i}{1+i}.$$

(β) §§ [21]-[35].

[illegible]).—§ [24]. It is assumed that the first payment of the annuity [illegible] made at the end of the year of death of (y), and that there will [illegible] proportionate payment at the death of (x).

[illegible]).—§§ [28]-[33].

CHAPTER XV.

(140).—§§ [6]–[11], [18], [19].
(141).—§ [8].
(142).—The value of the annuity is evidently equal to

$$\frac{1}{l_xl_yl_z}\int_1^{\infty} v^t.l_{z+t}.\mu_{z+t}.l_{x+t}[l_{y+t}.a_{x+t:\overline{y+t}}+(l_y-l_{y+t})a_{x+t}]dt$$

$$=\frac{1}{l_xl_yl_z}\int_1^{\infty} v^tl_{z+t}.\mu_{z+t}.l_{x+t}[l_{y+t}(a_{y+t}-a_{x+t:y+t})+l_y.a_{x+t}]dt.$$

This expression may be summed by one of the formulas of approximate summation in Chap. xxiv.

The annuity is also equal to that of (i) a reversionary annuity to (x), after (z); (ii) a reversionary annuity to (y) after the last survivor of (x) and (z) provided (z) fail first, that is

$$=a_{z|x}+a_{\overset{}{\underset{1}{xz}}|y}$$

$$=a_{z|x}+a^{\;1}_{xz|y}-a_{zxy}$$

$$=(a_x-a_{xyz})+a^{\;1}_{xz|y}.$$

The value of $a^{\;1}_{xz|y}$ may be obtained by reference to §§ [6]–[11], [18], [19].

(143).—§§ [12], [13].

CHAPTER XVI.

(144).—§§ [3]–[5], Chap. vii, § [14].
(145).—§ [6].
(146).—§§ [9], [10].

$$\frac{N_{x-t}}{D_x}=\frac{N_{x-t}}{D_{x-t}}.\frac{D_{x-t}}{D_x}=a_{x-t}.\frac{(1+i)^t}{{}_tp_{x-t}}.$$

From this expression it will be seen that $\frac{N_{x-t}}{D_x}$ represents the accumulated amount, after t years, of the value of an annuity on a life aged $(x-t)$, *allowing for mortality and interest.* If deduction be made of the accumulated amount of the payments made under the annuity (also allowing for mortality and interest)

$$=\frac{N_{x-t}-N_x}{D_{x-t}}\cdot\frac{D_{x-t}}{D_x},$$

the difference is equal to

$$\frac{N_{x-t}}{D_{x-t}}\cdot\frac{D_{x-t}}{D_x}-\frac{N_{x-t}-N_x}{D_{x-t}}\cdot\frac{D_{x-t}}{D_x}=\frac{N_x}{D_x}=a_x.$$

(147).—Chap. vii, § [48]. We have

$$(\mathbf{v}a)_x=\frac{kN_x+hS_{x+1}}{D_x}$$

$$(\mathbf{v}A)_x=\frac{kM_x+hR_{x+1}}{D_x}.$$

But $M_x=vN_{x-1}-N_x$; $R_{x+1}=vS_x-S_{x+1}$

$$\therefore\quad(\mathbf{v}A)_x=\frac{k(vN_{x-1}-N_x)+h(vS_x-S_{x+1})}{D_x}$$

$$=\frac{v(kN_{x-1}+hS_x)-(kN_x+hS_{x+1})}{D_x}.$$

$$=v\left((\mathbf{v}a)_{x-1}\frac{D_{x-1}}{D_x}\right)-(\mathbf{v}a)_x$$

$$=v\,(\mathbf{va})_x-(\mathbf{v}a)_x.$$

(148).—

$$\left(\frac{N_x}{D_x}+\frac{1}{4}\right)+\frac{D_{x+10}}{D_x}\left(\frac{N_{x+10}}{D_{x+10}}+\frac{1}{4}\right)+\frac{2D_{x+20}}{D_x}\left(\frac{N_{x+20}}{D_{x+20}}+\frac{1}{4}\right)$$

$$+\frac{4D_{x+30}}{D_x}\left(\frac{N_{x+30}}{D_{x+30}}+\frac{1}{4}\right)+\frac{8D_{x+40}}{D_x}\left(\frac{N_{x+40}}{D_{x+40}}+\frac{1}{4}\right)+\dots\dots$$

$$=\frac{1}{D_x}[N_x+N_{x+10}+2N_{x+20}+4N_{x+30}+8N_{x+40}+\dots\dots$$

$$+\tfrac{1}{4}(D_x+D_{x+10}+2D_{x+20}+4D_{x+30}+8D_{x+40}+\dots\dots)].$$

If $\Sigma N_x=(N_x+N_{x+10}+N_{x+20}+\dots\dots)$

and $\Sigma D_x=(D_x+D_{x+10}+D_{x+20}+\dots\dots)$,

the above becomes

$$\frac{1}{D_x}[\Sigma N_x+\Sigma N_{x+20}+2\Sigma N_{x+30}+4\Sigma N_{x+40}+\dots\dots$$

$$+\tfrac{1}{4}(\Sigma D_x+\Sigma D_{x+20}+2\Sigma D_{x+30}+4\Sigma D_{x+40}+\dots\dots)].$$

∴ The single premium is equal to

$$\frac{M_x+M_{x+1}+M_{x+2}+\dots\dots+M_{x+n-1}}{D_x}=\frac{R_x-R_{x+n}}{D_x};$$

the annual premium (α) payable for n years

$$=\frac{R_x-R_{x+n}}{N_{x-1}-N_{x+n-1}};$$

(β) payable throughout life

$$=\frac{R_x-R_{x+n}}{N_{x-1}}.$$

(150).— $$\frac{(M_x-M_{x+20})-\frac{1}{20}(R_{x+1}-R_{x+20}-19M_{x+20})}{(N_{x-1}-N_{x+19})-\frac{1}{20}(S_x-S_{x+19}-19N_{x+19})}$$

$$=\frac{M_x-\frac{1}{20}(R_{x+1}-R_{x+\text{[illegible]}})}{N_{x-1}-\frac{1}{20}(S_x-S_{x+20})}:$$

This may be also expressed as

$$v-\frac{N_x-\frac{1}{20}(S_{x+1}-S_{x+21})}{N_{x-1}-\frac{1}{20}(S_x-S_{x+20})},\quad\text{or}\quad\frac{D_x-\frac{1}{20}(N_x-N_{x+20})}{N_{x-1}-\frac{1}{20}(S_x-S_{x+20})}-d.$$

(151).—§§ [27]-[37].

(152).—§§ [39]-[43].

(153).—Equating the benefit and payment side, we have

$$N_{x+n}+\pi\{(1+j)C_x+[(1+j)+(1+j)^2]C_{x+1}+\ldots\ldots$$
$$+[(1+j)+(1+j)^2+\ldots\ldots+(1+j)^n]C_{x+n-1}\}$$

$=\pi(N_{x-1}-N_{x+n-1})$ [where C and N are calculated at the rate of interest i],

which becomes

$$N_{x+n}+\pi\{(1+j)M_x+(1+j)^2M_{x+1}+\ldots+(1+j)^nM_{x+n-1}$$
$$-(1+j)s'_{\overline{n}|}M_{x+n}\}$$

$$=\pi(N_{x-1}-N_{x+n-1})\quad\left[\text{where } s'_{\overline{n}|}=\frac{(1+j)^n-1}{j}\right],$$

whence

$$\pi=\frac{N_{x+n}}{(N_{x-1}-N_{x+n-1})-\{(1+j)M_x+(1+j)^2M_{x+1}+\ldots+(1+j)^nM_{x+n-1}-(1+j)s'_{\overline{n}|}M_{x+n}\}}$$

(154).—If $j=i$, the denominator in the above expression becomes

$$=(N_{x-1}-N_{x+n-1})-\{(1+i)(vN_{x-1}-N_x)+(1+i)^2(vN_x-N_{x+1})+\ldots$$
$$+(1+i)^n(vN_{x+n-2}-N_{x+n-1})-(1+i)s_{\overline{n}|}(vN_{x+n-1}-N_{x+n})\}$$

which becomes, after simplification and reduction,

$$=N_{x+n-1}(s_{\overline{n}|}+(1+i)^n-1)-N_{x+n}(1+i)s_{\overline{n}|}$$

$$=D_{x+n}(1+i)s_{\overline{n}|}$$

$$\therefore \quad \pi=\frac{N_{x+n}}{D_{x+n}(1+i)s_{\overline{n}|}}=\frac{via_{x+n}}{(1+i)^n-1}.$$

This result may also be expressed in the form $\frac{v^n a_{x+n}}{a_{\overline{n}|}}$, which is obviously equal to the annual premium for a deferred annuity when the element of mortality is entirely eliminated during the term of n years.

(155).—§§ [53], [54].

(156).—If π be the net annual premium, and $\pi(1+k)+c$ the corresponding office premium, we have

$$\pi\frac{N_{x-1}-N_{x+m+n-1}}{D_x}=\frac{D_{x+m+n}+M_{x+m}-M_{x+m+n}}{D_x}$$

$$+\frac{[\pi(1+k)+c](R_{x+1}-R_{x+m}-\overline{m-1}M_{x+m})}{D_x},$$

whence

$$\pi=\frac{D_{x+m+n}+M_{x+m}-M_{x+m+n}+c(R_{x+1}-R_{x+m}-\overline{m-1}M_{x+m})}{N_{x-1}-N_{x+m+n-1}-(1+k)(R_{x+1}-R_{x+m}-\overline{m-1}M_{x+m})}.$$

(157).—Making the same assumptions as in the previous solution, we have

$$\pi=\frac{M_x+c(R_x-R_{x+t})}{N_{x-1}-N_{x+t-1}-(1+k)(R_x-R_{x+t})}.$$

(158).— $$\pi\frac{N_{x-1}-N_{x+n-1}}{D_x}=\frac{M_x}{D_x}+\left(\pi-\frac{M_x}{N_{x-1}}\right)\left(\frac{R_x-R_{x+n}-nM_{x+n}}{D_x}\right)$$

whence

$$\pi=\frac{M_x-\frac{M_x}{N_{x-1}}(R_x-R_{x+n}-nM_{x+n})}{N_{x-1}-N_{x+n-1}-(R_x-R_{x+n}-nM_{x+n})}.$$

(159).—(*J.I.A.*, xxi, 67.)

(160).—§§ [92], [93].

CHAPTER XVII.

(161).—§ [5].

(162).—§§ [7]-[9]. $\frac{\mathrm{A}_x}{1-\mathrm{A}_y}$.

(163).—§§ [16]-[19]. $5\cdot\frac{\frac{1}{i}-(a_{61}+a_{50}+a_{45})}{1+a_7}$.

(164).—§§ [20]-[22].

(165).—Let x be the age of the present incumbent, s the annual income, κ the stipend of the curate-in-charge, f the expenditure at entry, and y the age at entry of the new incumbent: then the value of the nth presentation is equal to

$$\overline{\mathrm{A}}_x(\overline{\mathrm{A}}_y)^{n-1}\{(s-\kappa)\bar{a}_y-f\}\doteq\mathrm{A}_x\mathrm{A}_y{}^{n-1}\{(s-\kappa)(\mathring{a}_y+\tfrac{1}{2})-f\} \text{ approximately.}$$

(166).— $$\overline{\mathrm{A}}_x\{(s-\kappa)\bar{a}_y-f\}[(\overline{\mathrm{A}}_y)+(\overline{\mathrm{A}}_y)^4+(\overline{\mathrm{A}}_y)^7+\ldots]$$

$$=\text{approximately } \frac{\mathrm{A}_x\mathrm{A}_y}{1-(\mathrm{A}_y)^3}\{(s-\kappa)(\tfrac{1}{2}+\mathring{a}_y)-f\}.$$

CHAPTER XVIII.

(167).—Chap. xvi, §§ [14], [15]; Chap. xviii, §§ [4], [5].

(168).—If $\pi^{\;1}_{x\overline{n}|}$ be the net premium paid by the l_x persons, the accumulated amount paid (α) by the l_{x+n} survivors will, at the end of n years, be equal to

$$\pi^{\;1}_{x\overline{n}|}\{[(1+i)^n+(1+i)^{n-1}+\ldots+(1+i)]l_{x+n}\},$$

and the accumulated amount paid (β) by those dying in the several years will be equal to

$$\pi^{\;1}_{x\overline{n}|}\{(l_x-l_{x+1})(1+i)^n+(l_{x+1}-l_{x+2})[(1+i)^n+(1+i)^{n-1}]+\ldots\ldots$$
$$+(l_{x+n-2}-l_{x+n-1})[(1+i)^n+(1+i)^{n-1}+\ldots+(1+i)^2]$$
$$+(l_{x+n-1}-l_{x+n})[(1+i)^n+(1+i)^{n-1}+\ldots+(1+i)]\}.$$

The whole accumulation is thus equal to

$$\begin{aligned}
&\pi^{\;1}_{x\overline{n}|}\{(1+i)^n[(l_{x+n}+l_x+l_{x+1}+\ldots+l_{x+n-1})-(l_{x+1}+l_{x+2}+\ldots+l_{x+n})]\\
&\quad+(1+i)^{n-1}[(l_{x+n}+l_{x+1}+l_{x+2}+\ldots+l_{x+n-1})-(l_{x+2}+l_{x+3}+\ldots+l_{x+n})]\\
&\quad+\\
&\quad+(1+i)^2[(l_{x+n}+l_{x+n-2}+l_{x+n-1})-(l_{x+n-1}+l_{x+n})]\\
&\quad+(1+i)[(l_{x+n}+l_{x+n-1})-l_{x+n}]\\
&=\pi^{\;1}_{x\overline{n}|}\{(1+i)^n l_x+(1+i)^{n-1}l_{x+1}+\ \ldots\ +(1+i)^2 l_{x+n-2}+(1+i)l_{x+n-1}\}\\
&=\pi^{\;1}_{x\overline{n}|}\{(1+i)^{x+n}[\mathrm{D}_x+\mathrm{D}_{x+1}+\ \ldots\ +\mathrm{D}_{x+n-2}+\mathrm{D}_{x+n-1}]\}\\
&=\frac{\mathrm{D}_{x+n}}{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}}\cdot\frac{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}}{v^{x+n}}=l_{x+n}.
\end{aligned}$$

(169).—§§ [18], [22].

(170).—

$$\begin{aligned}
{}_n\mathrm{V}_x &> = < {}_{n-1}\mathrm{V}_{x+1}\\
\left(1-\frac{1+a_{x+n}}{1+a_x}\right) &> = < \left(1-\frac{1+a_{x+n}}{1+a_{x+1}}\right)\\
\frac{1+a_{x+n}}{1+a_{x+1}} &> = < \frac{1+a_{x+n}}{1+a_x}\\
1+a_x &> = < 1+a_{x+1}\\
a_x &> = < a_{x+1}.
\end{aligned}$$

As regards adult ages, at which alone policy-values are practically in question, the value of a_x decreases as age increases; but at the commencement and end of the mortality table the value of a_{x+1} is frequently greater than that of a_x, as in the Northampton Table from age 0 to 7, in the Carlisle Table from age 0 to 6, and 91 to 94, and in the Life Table in the *Text-Book* from age 0 to 4.

(171).—§ [17]. ${}_{n+m}\mathrm{P}_x(1+{}_{|m-1}a_{x+n})+d(1+a_{x+n})+{}_n\mathrm{V}'_x=1.$

(172).—

$$\begin{aligned}
\frac{\mathrm{P}_x-\mathrm{P}^{\;1}_{x\overline{n}|}}{\mathrm{P}^{\;1}_{x\overline{n}|}} &= \frac{\dfrac{\mathrm{M}_x}{\mathrm{N}_{x-1}}-\dfrac{\mathrm{M}_x-\mathrm{M}_{x+n}}{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}}}{\dfrac{\mathrm{D}_{x+n}}{\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1}}}\\
&=\frac{\mathrm{M}_x(\mathrm{N}_{x-1}-\mathrm{N}_{x+n-1})}{\mathrm{N}_{x-1}\cdot\mathrm{D}_{x+n}}-\frac{\mathrm{M}_x-\mathrm{M}_{x+n}}{\mathrm{D}_{x+n}}\\
&=\frac{\mathrm{M}_{x+n}}{\mathrm{D}_{x+n}}-\frac{\mathrm{M}_x}{\mathrm{N}_{x-1}}\cdot\frac{\mathrm{N}_{x+n-1}}{\mathrm{D}_{x+n}}=\mathrm{A}_{x+n}-\mathrm{P}_x(1+a_{x+n})\\
&={}_n\mathrm{V}_x.
\end{aligned}$$

This relation may be proved also from the following considerations:—An annual payment of P_x for n years may be regarded as made up of (i) the annual premium to cover the risk of death during the term $= P^{1}_{x\overline{n}|}$; (ii) the annual premium to secure a sum equal to the value of the policy at the end of the term, if the life be then in existence $= {}_nV_x P_{x\overline{n}|}^{\;\;1}$; that is,

$$P_x = P^{1}_{x\overline{n}|} + {}_nV_x P_{x\overline{n}|}^{\;\;1}$$

whence
$$ {}_nV_x = \frac{P_x - P^{1}_{x\overline{n}|}}{P_{x\overline{n}|}^{\;\;1}}.$$

The formula $\dfrac{P_x - P^{1}_{x\overline{n}|}}{P_{x\overline{n}|}^{\;\;1}}$ also exhibits the value of a policy, effected on a life (x) after n years, as the accumulation, allowing for mortality and interest, of the amount paid in excess of that required each year for the actual risk of death.

(173).—§§ [38]–[44].

(174).—§§ [49]–[55].

(175).—§§ [51], [56].

(176).—§ [69]. We have

$$\text{(A)}\quad {}_nV_x = \frac{a_x - a_{x+n}}{1+a_x} = \frac{(vp_x + v^2{}_2p_x + \ldots) - (vp_{x+n} + v^2{}_2p_{x+n} + \ldots)}{1 + vp_x + v^2{}_2p_x + \ldots}.$$

$$\text{(B)}\quad {}_nV'_x = \frac{a'_x - a'_{x+n}}{1+a'_x} = \frac{(vcp_x + v^2c^2{}_2p_x + \ldots) - (vcp_{x+n} + v^2c^2{}_2p_{x+n} + \ldots)}{1 + vcp_x + v^2c^2{}_2p_x + \ldots}.$$

In the second equation vc takes the place of v in the first equation throughout: it is therefore evident that the policy-values by Table (B) will be equal to those under Table (A) when the rates of interest adopted are such that the present value of 1 due a year hence equals v in the one case and vc in the other; that is, the policy-values by Table (B), computed at the rate of interest i, will be equal to the policy-values by Table (A), computed at the rate of interest $\left(\dfrac{1+i}{c} - 1\right)$.

(177).—From §§ [45]–[68], and especially from Tables F, G, and H, it will be seen that no conclusion as to the relative policy-values at different ages and durations can be arrived at from the fact that the rate of mortality in one table is throughout greater than that in another table.

(178).—§§ [76]–[79].

(179).—§ [81]. The modified formula would be

$$\Sigma A_{x+n} - \Sigma P_x(\tfrac{1}{2} + a_{x+n}).$$

(180).—§§ [94]-[100].

(181).—

$$A_{x+n.\overline{m}|} - P_{x.\overline{n+m}|}(1 + {}_{|m-1}a_{x+n}) = (P_{x+n.\overline{m}|} - P_{x.\overline{n+m}|})(1 + {}_{|m-1}a_{x+n})$$

$$= \frac{P_{x+n.\overline{m}|} - P_{x.\overline{n+m}|}}{P_{x+n\ \overline{m}|} + d}.$$

We have also

$$\frac{P_{x+n\ \overline{m}|} - P_{x.\overline{n+m}|}}{P_{x+n\ \overline{m}|} + d} = \frac{\dfrac{1}{1 + {}_{|m-1}a_{x+n}} - d - \dfrac{1}{1 + {}_{|n+m-1}a_x} + d}{\dfrac{1}{1 + {}_{|m-1}a_{x+n}}}$$

$$= \frac{1 + {}_{|m-1}a_{x+n}}{1 + {}_{|m-1}a_{x+n}} - \frac{1 + {}_{|m-1}a_{x+n}}{1 + {}_{|n+m-1}a_x} = \frac{(1 + {}_{|n+m-1}a_x) - (1 + {}_{|m-1}a_{x+n})}{1 + {}_{|n+m-1}a_x}$$

$$= \frac{\dfrac{1 - A_{x\ \overline{n+m}|}}{d} - \dfrac{1 - A_{x+n\ \overline{m}|}}{d}}{\dfrac{1 - A_{x.\overline{n+m}|}}{d}} = \frac{A_{x+n\ \overline{m}|} - A_{x\ \overline{n+m}|}}{1 - A_{x.\overline{n+m}|}}$$

(182).—§§ [108]-[112].

The value of the endowment policy after n years, by the *retrospective* method, is equal to the accumulated net premiums, less the accumulated claims in respect of premiums returnable at death, that is,

$$\pi\frac{N_{x-1} - N_{x+n-1}}{D_{x+n}} - [\pi(1+\kappa) + c]\frac{R_x - R_{x+n} - nM_{x+n}}{D_{x+n}}$$

The value by the *prospective* method will be

$$= \frac{D_{x+n+m}}{D_{x+n}} + [\pi(1+\kappa) + c]\frac{n(M_{x+n} - M_{x+n+m}) + R_{x+n} - R_{x+n+m} - mM_{x+n+m}}{D_{x+n}}$$

$$-\pi\cdot\frac{N_{x+n-1} - N_{x+n+m-1}}{D_{x+n}}.$$

If the identity of these two expressions be assumed, and the value of π deduced therefrom, it will be found that

$$\pi = \frac{D_{x+n+m} + c[R_x - R_{x+n+m} - (n+m)M_{x+n+m}]}{N_{x-1} - N_{x+n+m-1} - (1+\kappa)(R_x - R_{x+n+m} - (n+m)M_{x+n+m})}$$

This being the correct expression for π (see Chap. xvi, formula 47) the identity of the above expressions is established.

(183).—§§ [111]-[113].

(184).—§§ [19], [122]-[124].

(185).—§§ [129]-[134].

(186).—It would be necessary in the first instance to ascertain the net premium to be valued, and to do this some assumption must be made as to the loading; *e.g.*, if we assume this to consist of m per-cent upon the gross premium and a constant of c per-cent, we have the following equation to determine the net premium

$$100\text{M}_{41}=\frac{100-m}{100}\{2{\cdot}258\text{N}_{40}+1{\cdot}229(\text{N}_{47}+\text{N}_{54})\}-c\text{N}_{40}.$$

Hence, the net premium after the second seven years will be

$$\left(\frac{100-m}{100}4{\cdot}717-c\right)=\text{P}'\text{ say,}$$

and the value of the policy, the age of the life being now 69¾, and the next annual premium due in three months' time would be

$${}_{28\frac{3}{4}}\text{V}'_{41}=\text{A}_{69\frac{3}{4}}-(\tfrac{3}{4}+a_{69\frac{3}{4}})\text{P}'.$$

(187).—§§ [14], [15].

(1) The premiums must be net, without expenses, or the loading exactly absorbed by expenses.
(2) The premiums must be payable at the beginning of each year.
(3) The rate of interest assumed must be identical with that realized.
(4) The rate of mortality assumed must be identical with that experienced.
(5) The claims must all be payable at the end of the year.

Assuming that l_x policies were effected n years ago at age x, the following equation exhibits the relation stated in the question:—

$$l_{x+n-1}[{}_{n-1}\text{V}_x(1+i)+\text{P}_x(1+i)]=l_{x+n}\cdot{}_{n}\text{V}_x+d_{x+n-\text{[illegible]}}$$

or, dividing throughout by l_{x+n-1},

$${}_{n-1}\text{V}_x(1+i)+\text{P}_x(1+i)=p_{x+n-1}\cdot{}_{n}\text{V}_x+q_{x+n-1},$$

which may be written

$${}_{n-1}\text{V}_x+\text{P}_x=v(q_{x+n-1}+p_{x+n-1}\cdot{}_{n}\text{V}_x).$$

In the case where the premiums are payable at intervals throughout the year, the above expressions will be modified as follows:—Making the usual assumptions that a group of lives entering in the course of the nth year prior to the date of valuation at office age (x), were on the average of the precise age $(x-\frac{1}{2})$ at entry, entered on the average $(n-\frac{1}{2})$ years ago, and completed the average age $(x+n)$ at the date of valuation, we have (assuming yearly payments throughout) the following expressions for the value of the policies a year ago and at the present time respectively,

$$
\begin{aligned}
{}_{n-1\frac{1}{2}}V_{x-\frac{1}{2}} &= A_{x+n-1} - P_{x-\frac{1}{2}}(\tfrac{1}{2}+a_{x+n-1}) \\
&= \left(1-\frac{1+a_{x+n-1}}{1+a_{x-\frac{1}{2}}}+\frac{P_{x-\frac{1}{2}}}{2}\right) \\
{}_{n-\frac{1}{2}}V_{x-\frac{1}{2}} &= A_{x+n} - P_{x-\frac{1}{2}}(\tfrac{1}{2}+a_{x+n}) \\
&= \left(1-\frac{1+a_{x+n}}{1+a_{x-\frac{1}{2}}}+\frac{P_{x-\frac{1}{2}}}{2}\right).
\end{aligned}
$$

whence, from the relations,

$$1+a_{x+n-1}=1+vp_{x+n-1}(1+a_{x+n})$$

and

$$P_{x-\frac{1}{2}}=\left(\frac{1}{1+a_{x-\frac{1}{2}}}-d\right)$$

we can obtain the expression

$$\left[{}_{n-1\frac{1}{2}}V_{x-\frac{1}{2}}+P_{x-\frac{1}{2}}\left(\frac{1+vp_{x+n-1}}{2}\right)\right]=[vp_{x+n-1}({}_{n-\frac{1}{2}}V_{x-\frac{1}{2}})+vq_{x+n-1}],$$

showing the desired relation.

CHAPTER XIX.

(188).—We have

$${}_{|n}a_x=\frac{1}{P_{x:\overline{n+1}|}+d}-1.$$

The assurance required is therefore an endowment assurance on (x), payable at age $(x+n+1)$ or at previous death. The sum assured is equal to

$$\frac{1}{P_{x:\overline{n+1}|}+d},$$

and the annual premium to

$$\frac{P_{x:\overline{n+1}|}}{P_{x:\overline{n+1}|}+d}.$$

The total outlay is made up of the first premium and the value of the life interest

$$=\frac{P_{x:\overline{n+1}|}}{P_{x:\overline{n+1}|}+d}+\frac{1}{P_{x:\overline{n+1}|}+d}-1=\frac{1-d}{P_{x:\overline{n+1}|}+d}.$$

The life interest provides

$$\frac{P_{x:\overline{n+1}|}}{P_{x:\overline{n+1}|}+d}$$

as the annual premium, and

$$\frac{d}{P_{x:\overline{n+1}|}+d}$$

as interest on the total outlay, the sum of these two expressions making up unity, the amount of the annuity.

The sum assured provides the return of the total outlay, with interest thereon for the last year.

(189).—§§ [3], [10]. The value of the life interest, the age of the life tenant being (x), is equal to

$$s\left(\frac{1}{P_x+d}-1\right).$$

The sum to be assured is equal to $\frac{s}{P_x+d}$, and the value of the policy after t years is equal to $\frac{s}{P_x+d}[1-(P_x+d)(1+a_{x+t})]$

$$=s\left[\frac{1}{P_x+d}-(1+a_{x+t})\right].$$

The value of the life interest at the end of the t years is $=(s\times a_{x+t})$.

The value of the policy and of the life interest together is therefore $=s\left(\frac{1}{P_x+d}-1\right)$, a constant value irrespective of t, and equal to the value of the life interest at the commencement.

If the value of the life interest and of the policy, after t years, be computed upon *office* rates of premium throughout, we shall have

Market value of the life interest after t years

$$=s\left(\frac{1}{P_{x+t}+d}-1\right).$$

Value of the policy after t years, computed by the "re-assurance" or "hypothetical" method (Chap. xviii, §§ [33]-[36])

$$=\frac{s}{P_x+d}\left(1-\frac{P_x+d}{P_{x+t}+d}\right)=s\left(\frac{1}{P_x+d}-\frac{1}{P_{x+t}+d}\right).$$

The sum of these two values being, as before,

$$=s\left(\frac{1}{P_x+d}-1\right).$$

(190).—We have

$$a'_x=\frac{1}{P_x+d}-1$$

Sum assured $\qquad =\frac{1}{P_x+d}=1+a'_x=S.$

Value of policy after n years

$$=S\{1-(P_x+d)(1+a_{x+n})\}$$

$$=S\left\{1-\frac{1+a_{x+n}}{1+a'_x}\right\}$$

$$=(1+a'_x)\left\{1-\frac{1+a_{x+n}}{1+a'_x}\right\}$$

$$=a'_x-a_{x+n}.$$

(191).—§ [28].

Since the value of a reversionary life interest of 1 is equal to

$$\frac{1-(P_x+d)(1+a_{xy})}{P_x+d},$$

an advance of 1 will represent the value of a life interest of

$$\frac{P_x+d}{1-(P_x+d)(1+a_{xy})}.$$

The sum assured for a reversionary life interest of 1 is equal to $\frac{1}{P_x+d}$;

$\therefore$ the sum assured for a reversionary life interest of $\frac{P_x+d}{1-(P_x+d)(1+a_{xy})}$ is equal to $\frac{1}{1-(P_x+d)(1+a_{xy})}$.

The cost of the annuity purchased during the joint lives is evidently

$$\frac{P_x+d}{1-(P_x+d)(1+a_{xy})}\cdot(a_{xy}).$$

The total outlay is made up of (1) the value given for the reversionary life interest$=1$; (2) the cost of the annuity purchased during the joint lives $=\dfrac{(P_x+d)(a_{xy})}{1-(P_x+d)(1+a_{xy})}$; (3) the first premium for the assurance $=\dfrac{P_x}{1-(P_x+d)(1+a_{xy})}$, the sum of these values being equal to $\dfrac{1-d}{1-(P_x+d)(1+a_{xy})}$, the total outlay.

The annuity will provide (1) the premium for the assurance $=\dfrac{P_x}{1-(P_x+d)(1+a_{xy})}$; (2) the interest upon the total outlay $=\dfrac{d}{1-(P_x+d)(1+a_{xy})}$.

The sum assured will provide the return of the total outlay, with one year's interest thereon, at the end of the year of the failure of the life (x).

Let the age of (x) at the end of the year of death of (y) be $=(x+n)$; the value of the annuity is then equal to

$$\frac{(P_x+d)a_{x+n}}{1-(P_x+d)(1+a_{xy})},$$

and the value of the policy is then equal to

$$\frac{1-(P_x+d)(1+a_{x+n})}{1-(P_x+d)(1+a_{xy})},$$

the sum of these values being equal to

$$\frac{1-(P_x+d)}{1-(P_x+d)(1+a_{xy})},$$

an expression independent of n, and which, when put in the form $\left(\dfrac{1}{P_x+d}-1\right)\left(\dfrac{P_x+d}{1-(P_x+d)(1+a_{xy})}\right)$, will be seen to represent the value of an *immediate* life interest on (x), the amount of the life interest being equal to that involved in the question. (*J.I.A.*, xiv, 426-7.)

(192).—§§ [26], [27].

(193).—The net value is $=a_x-a_{x:\overline{ys}}$, or in market values, and allowing for whole-world and issue premiums,

$$\frac{1-\dfrac{m+n}{100}}{P_x+d}-(1+a_{x\cdot\overline{ys}})$$

(194).—§§[41]-[45].

CHAPTER XX.

(195).—The value of the sick allowance to age 70 is (§ [5])

$$=_{|70-x}s_x=\frac{D_xv^{\frac{1}{2}}z_x+D_{x+1}v^{\frac{1}{2}}z_{x+1}+\ldots\ldots+D_{69}v^{\frac{1}{2}}z_{69}}{D_x}.$$

But $s_x=$ by hypothesis $A+Bq_x$, whence we have

$$_{|70-x}s_x=\frac{Av^{\frac{1}{2}}}{D_x}(D_x+D_{x+1}+\ldots\ldots+D_{69})$$

$$+\frac{Bv^{\frac{1}{2}}}{D_x}(D_xq_x+D_{x+1}q_{x+1}+\ldots\ldots+D_{69}q_{69})$$

$$=A\frac{\bar{N}_x-\bar{N}_{70}}{D_x}+B\frac{\bar{M}_x-\bar{M}_{70}}{D_x}$$

$$=A_{|70-x}\bar{a}_x+B_{|70-x}\bar{A}_x$$

$$=B\left(\frac{D_x-D_{70}}{D_x}\right)+(A-\delta B)\left(a_x-\frac{D_{70}}{D_x}\bar{a}_{70}\right).$$

(196).—We have, by Table (A),

$$s_x=v^{\frac{1}{2}}[z_x+vp_xz_{x+1}+v^2{}_2p_xz_{x+2}+\ldots\ldots]$$

and, by Table (B),

$$s^1_x=v^{\frac{1}{2}}[z_x+v(1-\kappa)p_xz_{x+1}+v^2(1-\kappa)^2{}_2p_xz_{x+2}+\ldots\ldots].$$

Comparing the expressions within brackets [ ], we see that v, in the first formula, is throughout replaced by $v(1-\kappa)$ in the second formula; therefore the value of $s^1_x(1-\kappa)^{\frac{1}{2}}$ by Table (B), where s^1_x is computed at the rate i, is equal, for all values of x, to the value of s_x by Table (A), computed at the rate $\left(\frac{1+i}{1-\kappa}-1\right)$.

(197).—If at any given age x there be d_x deaths, there will by hypothesis be $2d_x$ persons constantly sick throughout the year; that is, there will be $\frac{2\times365\frac{1}{4}}{7}d_x$ weeks' sickness; the cost of which will be at 10*s*. per week $=\frac{365\frac{1}{4}}{7}d_x=52{\cdot}18d_x$, and the value of the sick allowance to age 65 will be $=\Sigma_x^{64}52{\cdot}18d_x=52{\cdot}18\left(\frac{\bar{M}_x-\bar{M}_{65}}{D_x}\right)$

(198).—§§ [1]-[5]. (*J.I.A.*, xxvii, 280, 281.)

$$\left.\begin{array}{ll}(\alpha) & s_x=\dfrac{K_x}{D_x}\\[2ex] (\beta) & {}_{|70-x}s_x=\dfrac{K_x-K_{70}}{D_x}\\[2ex] (\gamma) & {}_{60-x|}s_x=\dfrac{K_{60}}{D_x}\end{array}\right\}\text{ where } K_x=v^{x+\frac{1}{2}}(lz)_x+v^{x+1\frac{1}{2}}(lz)_{x+1}+\ldots\ldots$$

(199).—§ [8]. The formula (for the present value of an allowance c 1 per week) becomes

$$={}_{|70-x}s_x+{}_{70-x|}\bar{a}_x=\frac{(K_x-K_{70})+52\bar{N}_{70}}{D_x}.$$

(200).—(*J.I.A.*, xxvii, 281.)

$$c_x=\frac{(K^1_x-K^1_{70})+\frac{1}{2}(K^2_x-K^2_{70})+\frac{1}{4}(K^{3+}_x-K^3_{70})+13\bar{N}_{70}+10\bar{M}_x}{\bar{N}_x-\bar{N}_{70}}$$

and the weekly contribution, loaded for expenses, is $=\dfrac{c_x}{\cdot 7\times 52}$. Th reserves after n years are equal to

$$\frac{10M_{x+n}+(K^1_{x+n}-K_{70})+\frac{1}{2}(K^2_{x+n}-K^2_{70})+\frac{1}{4}(K^3_{x+n}-K^3_{70})+13N_{70}}{D_{x+n}}$$

$$-\frac{c_x(N_{x+n}-N_{70})}{D_{x+n}}.$$

CHAPTER XXI.

(201).—§§ [43], [44], [101].

(202).—§§ [49], [52].

(203).—§§ [54], [57]-[60].

(204).—§§ [57]-[59].

(205).—Gray's *Tables and Formulæ*,* Chap. vi, §§ (193), (222)

LEMMA. If B denote the present value of a benefit of 1 upon a give life or combination of lives, and such that, in the case of a combinatior of lives the risk is determined by the failure of any one of them; and i: B_1 denote the present value of a similar benefit on a life or combinatior of lives respectively one year older than those on which B depends; if moreover, p denote the probability of a payment of B being received in the first year, and π the probability of the single life, or of all the lives on which that benefit depends, surviving a year;—then will the following equation always subsist:—

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* This Treatise being out of print, and practically inaccessible to students, th solutions to problems (205) and (206) have been extracted in full.

$$B = v\pi\left(\frac{p}{\pi} + B_1\right).$$

For, if the benefit makes its payments at the end of the year in which they respectively become due (as is always the case unless it be otherwise expressly stated) the value in respect of the first year is obviously vp. And the value in respect of the years following the first is $v\pi B_1$; for B_1 is the value, *at the time it is entered upon*, of the remaining portion of the benefit, π is the probability of its being entered upon, and $v\left(=\frac{1}{1+i}\right)$ is the ratio in which the value is diminished on account of the time that has to elapse before it is realized. Hence, a whole being equal to the sum of its parts,

$$B = vp + v\pi B_1$$

or

$$B = v\pi\left(\frac{p}{\pi} + B_1\right).$$

In this expression we have, for the benefit A_x,

$$\pi = p_x, \qquad B = A_x$$

$$p = (1 - p_x), \qquad B_1 = A_{x+1}$$

$$\therefore \quad A_x = vp_x\left(\frac{1-p_x}{p_x} + A_{x+1}\right) = vp_x[(p_x^{-1} - 1) + A_{x+1}].$$

(206).—Gray's *Tables and Formulæ*,* Chap. vi, §§ (194), (197), (198). The benefits to which the demonstration applies and in regard to which, consequently, the equation deduced subsists, must fulfil these two conditions: *First*, that the payment in respect of any one year in which it may become due shall be always 1. The fulfilment of this condition obviously precludes the application of the formula to increasing or decreasing benefits. *Secondly*, the benefit must be such that in the case of two or more lives the risk will be determined by the first failure that takes place from amongst those lives. The fulfilment of this condition excludes from the application of the formula benefits depending upon specified orders of survivorship amongst three or more. live *Thirdly*, the benefit B_1, it is stated in the Lemma, must be "similar" t the benefit B. The only restriction here implied, beyond that of it being of the same amount, and subject to like conditions in respect o the life or lives on which it depends as B_1, is, that its duration sha extend to the same period of life: in other words, that both benefit shall cease at the same age. It is no matter what that age is, whethe the limiting age of the table or any less tabular age; the formula wi still apply, provided only the age of cessation in respect of both benefi is the same. The formula consequently holds in the case of temporar as well as whole-life benefits. *Fourthly*, the formula holds also in th case of deferred benefits. To adapt it to such it is only necessary t make $p=0$ for those years in which no payment is made. This reduce the formula, in the case of deferred benefits, to $B=v\pi B_1$.

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* This Treatise being out of print, and practically inaccessible to students, the solutions to problems (205) and (206) have been extracted in full.

The following formulas show the application of the Lemma to th case of the benefits most frequently required:—

$$a_x=vp_x(1+a_{x+1}), \qquad {}_{|n}a_x=vp_x(1+{}_{|n-1}a_{x+1})$$

$$_{n|}a_x=vp_x\times{}_{n-1|}a_{x+1}, \qquad {}_{n|t}a_x=vp_x\times{}_{n-1|t}a_{x+1}$$

$$A_x=vp_x[(p_x^{-1}-1)+A_{x+1}], \qquad {}_{|n}A_x=vp_x[(p_x^{-1}-1)+{}_{|n-1}A_{x+1}]$$

$$_{n|}A_x=vp_x\times{}_{n-1|}A_{x+1}, \qquad A_{x\overline{n|}}=vp_x[(p_x^{-1}-1)+A_{x+1.\overline{n-1|}}]$$

(207).—§ [69].
(208).—§§ [72]-[81].

(209).—Chap. xviii, § [17].

$$\begin{aligned}{}_nV_x&=A_{x+n}-\pi_x(1+a_{x+n})\\&=(\pi_{x+n}-\pi_x)(1+a_{x+n})\\&=\left(\frac{1}{1+a_{x+n}}-d-\pi_x\right)(1+a_{x+n})\\&=1-(\pi_x+d)(1+a_{x+n}).\end{aligned}$$

(210).—§ [82].
(211).—§§ [95]-[98].
(212).—§§ [99]-[101].
(213).—*Institute of Actuaries' Life Tables* (Introduction, p lviii, lxviii)

(214).—By the formula

$$a_x = vp_x(1+a_{x+1})$$

a table of values of a_x can be computed. We then have

$$\bar{a}_x = a_x + \frac{1}{2} - \frac{\mu_x+\delta}{12}$$

$$\bar{a}_{x+1} = a_{x+1} + \frac{1}{2} - \frac{\mu_{x+1}+\delta}{12}$$

$$\Delta\bar{a}_x = \Delta a_x - \frac{\Delta\mu_x}{12}.$$

from which the continuous construction of the values of $\bar{a}_x$ can be completed.

A table of values of $\bar{A}_x$ can be constructed by the formulas

$$\bar{A}_x = 1-\delta\bar{a}_x$$

$$\bar{A}_{x+1} = 1-\delta\bar{a}_{x+1}$$

$$\Delta\bar{A}_x = -\delta\Delta\bar{a}_x.$$

(215).— $$A^{1}_{x\overline{1|}} = vq_x$$

$$A^{1}_{x+1.\overline{1|}} = vq_{x+1}$$

$$\Delta A^{1}_{x.\overline{1|}} = v\Delta q_x = -v\Delta p_x.$$

CHAPTER XXII.

(216).—§ [10].
(217).—§ [10].
(218).—§ [22].
(219).—§ [11]. Sunderland's *Notes on Finite Differences* (Chap. iii, § 3).

(220).—(a) § [21]. $u_n = u_0 + n\Delta u_0 + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_0 + \dots$

Writing u_{x+n} for u_n, and u_x for u_0, we have

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{\underline{|2}}\Delta^2 u_x + \dots$$

(β) § [18]. $\Delta^n u_0 = u_n - n u_{n-1} + \frac{n(n-1)}{|\underline{2}} u_{n-2} - \dots$

Writing u_x for u_0, and u_{x+n} for u_n, we have

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{|\underline{2}} u_{x+n-2} - \dots$$

(221).—§ [18]. If second differences are constant, $\Delta^3 u_x = 0$; but

$$\Delta^3 u_x = \Delta^2 u_{x+1} - \Delta^2 u_x = \Delta u_{x+2} - 2\Delta u_{x+1} + \Delta u_x = 0.$$

(222).—We have $u_x = u_x$

$$u_{x+a} = u_x + \Delta u_x$$

$$u_{x+2a} = u_x + \Delta u_x + \Delta(u_x + \Delta u_x)$$

$$= u_x + 2\Delta u_x + \Delta^2 u_x$$

$$u_{x+3a} = u_x + 2\Delta u_x + \Delta^2 u_x$$

$$+ \Delta(u_x + 2\Delta u_x + \Delta^2 u_x)$$

$$= u_x + 3\Delta u_x + 3\Delta^2 u_x + \Delta^3 u_x,$$

and evidently $u_{x+na} = u_x + n\Delta u_x + \frac{n(n-1)}{|\underline{2}} \Delta^2 u_x$

$$+ \frac{n(n-1)(n-2)}{|\underline{3}} \Delta^3 u_x + \dots$$

Writing now n for na, and $\frac{n}{a}$ for n, we have

$$u_{x+n} = u_x + \frac{n}{a}\Delta u_x + \frac{\frac{n}{a}\left(\frac{n}{a} - 1\right)}{|\underline{2}} \Delta^2 u_x$$

$$+ \frac{\frac{n}{a}\left(\frac{n}{a} - 1\right)\left(\frac{n}{a} - 2\right)}{|\underline{3}} \Delta^3 u_x + \dots$$

$$= u_x + \frac{n}{a}\Delta u_x + \frac{n(n-a)}{|\underline{2}a^2} \Delta^2 u_x$$

$$\frac{n(n-a)(n-2a)}{3a^3} \Delta^3 u_x + \dots$$

(223).—§ [21]. The required formula is

$$u_{x+m}=u_x+m\Delta u_x+\frac{m(m-1)}{\underline{|2}}\Delta^2u_x$$
$$+\frac{m(m-1)(m-2)}{\underline{|3}}\Delta^3u_x+\frac{m(m-1)(m-2)(m-3)}{\underline{|4}}\Delta^4u_x+\&c.$$

Then we have, if $x=1$, $m=(x-1)$,

$$u_x=u_1+(x-1)\Delta u_1+\frac{(x-1)(x-2)}{\underline{|2}}\Delta^2u_1$$
$$+\frac{(x-1)(x-2)(x-3)}{\underline{|3}}\Delta^3u_1+\frac{(x-1)(x-2)(x-3)(x-4)}{\underline{|4}}\Delta^4u_1$$

From the values given in the question, we have $u_1=4$, $\Delta u_1=26$, $\Delta^2u_1=64$, $\Delta^3u_1=66$, $\Delta^4u_1=24$; and inserting these values, and reducing, the above formula becomes

$$u_x=4+(x-1)26+\frac{(x-1)(x-2)}{\underline{|2}}64+\frac{(x-1)(x-2)(x-3)}{\underline{|3}}66$$
$$+\frac{(x-1)(x-2)(x-3)(x-4)}{\underline{|4}}24$$
$$=x^4+x^3+x^2+x.$$

(224).—Chap. xxiii, § [13]. From the formula

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{\underline{2}}\Delta^2u_x+\frac{n(n-1)(n-2)}{\underline{|3}}\Delta^3u_x+\dots$$

we have, when $n=\frac{t}{n}$,

$$u_{x+\frac{t}{n}}=u_x+\frac{t}{n}\Delta u_x+\frac{\frac{t}{n}\left(\frac{t}{n}-1\right)}{\underline{|2}}\Delta^2u_x$$
$$+\frac{\frac{t}{n}\left(\frac{t}{n}-1\right)\left(\frac{t}{n}-2\right)}{\underline{|3}}\Delta^3u_x+\dots$$
$$=u_x+\frac{t}{n}\Delta u_x+\frac{t(t-n)}{n^2\underline{|2}}\Delta^2u_x$$
$$+\frac{t(t-n)(t-2n)}{n^3\underline{|3}}\Delta^3u_x+\dots$$

(225).— (α) From Example (222), putting h for a, we have

$$u_{x+n}=u_x+\frac{n}{h}\Delta u_x+\frac{n(n-h)}{\underline{|2}\,h^2}\Delta^2 u_x$$

$$+\frac{n(n-h)(n-2h)}{\underline{|3}\,h^3}\Delta^3 u_x+\ldots$$

Putting now $x=0$, $n=x$, we have

$$u_x=u_0+\frac{x}{h}\Delta u_0+\frac{x(x-h)}{\underline{|2}\,h^2}\Delta^2 u_0$$

$$+\frac{x(x-h)(x-2h)}{\underline{|3}\,h^3}\Delta^3 u_0+\ldots$$

(β) Inserting the values given in the question,

$$u_x=0+\frac{x}{h}+\frac{x(x-h)}{2h^2}14$$

$$+\frac{x(x-h)(x-2h)}{6h^3}36+\frac{x(x-h)(x-2h)(x-3h)}{24h^4}.$$

$$=\left(\frac{x}{h}\right)^4.$$

(226).—The common difference of x being $=h$,

$$(x+n)^5=u_{x+n}=u_x+\frac{n}{h}\Delta u_x+\frac{n(n-h)}{2h^2}\Delta^2 u_x+\&c.$$

$$=u_x+\frac{n}{\underline{|1}}\frac{\Delta u_x}{\Delta x}+\frac{n(n-h)}{\underline{|2}}\cdot\frac{\Delta^2 u_x}{(\Delta x)^2}+\ldots$$

If Δx, and therefore h, be infinitely small, the above expressi becomes

$$u_{x+n}=u_x+\frac{n}{\underline{|1}}\frac{du_x}{dx}+\frac{n^2}{\underline{|2}}\frac{d^2u_x}{dx^2}+\ldots$$

where $\frac{du_x}{dx}$; $\frac{d^2u_x}{dx^2}$; &c., are successive differential coefficients of *i.e.*, of x^5. In this case,

$$\frac{du_x}{dx}=5x^4;\quad \frac{d^2u_x}{dx^2}=20x^3;$$

$$\frac{d^3u_x}{dx^3}=60x^2;\quad \frac{d^4u_x}{dx^4}=120x;\quad \frac{d^5u_x}{dx^5}=120.$$

Substituting these values,

$$u_{x+n}=x^5+5nx^4+10n^2x^3+10n^3x^2+5n^4x+n^5$$
$$=(x+n)^5.$$

(227).—Sunderland's *Notes on Finite Differences*, Chap. i, § 6, (2) (β)

$$\Delta(u_xv_x)=u_x\Delta v_x+v_x\Delta u_x+\Delta u_x\Delta v_x.$$

Repeating the operation of Δ on each term,

$$\Delta^2(u_xv_x)=u_x\Delta^2v_x+2\Delta u_x(\Delta v_x+\Delta^2v_x)$$
$$+\Delta^2u_x(v_x+2\Delta v_x+\Delta^2v_x).$$

If $u_x=x$, $v_x=\log x$, this becomes

$$=\Delta^2(x\log x)=x\Delta^2\log x+2(\Delta\log x+\Delta^2\log x).$$

(228).— $$u_x=b_1x+b_2x^2+b_3x^3+\ .\ .$$

Writing now $$b_2=b_1+\Delta b_1=(1+\Delta)b_1$$
$$b_3=b_1+2\Delta b_1+\Delta^2b_1=(1+\Delta)^2b_1$$
&c. &c. &c.

$$u_x=b_1x+(1+\Delta)b_1x^2+(1+\Delta)^2b_1x^3+\ .\ .\ .\ .$$
$$=[x+(1+\Delta)x^2+(1+\Delta)^2x^3+\ .\ .\ .\ .\ .]b_1$$
$$=\frac{x}{1-x(1+\Delta)}\cdot b_1=\frac{x}{(1-x)-x\Delta}\cdot b_1$$
$$=x\{(1-x)-x\Delta\}^{-1}.b_1$$
$$=x\{(1-x)^{-1}+(1-x)^{-2}x\Delta+(1-x)^{-3}(x\Delta)^2+\ .\ .\ .\ .\ .\}b_1$$
$$=\frac{xb_1}{1-x}+\frac{x^2\Delta b_1}{(1-x)^2}+\frac{x^3\Delta^2b_1}{(1-x)^3}+\ .\ .\ .$$

(229).—We have, by successive substitutions,

$$u_n=u_{n-1}+\Delta u_{n-1}$$
$$=u_{n-1}+\Delta u_{n-2}+\Delta^2u_{n-2}$$
$$=u_{n-1}+\Delta u_{n-2}+\Delta^2u_{n-3}+\Delta^3u_{n-3},$$

and ultimately, continuing the same process,

$$u_n=(u_{n-1}+\Delta u_{n-2}+\Delta^2u_{n-3}+\Delta^3u_{n-4}+\ .\ .\ .\ .\ .\ +\Delta^{n-2}u_1)+\Delta^{n-1}u_1.$$

Let $$u_1=a$$
$$u_2=u_1+\Delta u_1=2\Delta u_1$$ by question;

whence $\Delta u_1 = u_1 = a$ and $u_2 = 2\Delta u_1 = 2a$.

Similarly $u_3 = u_2 + \Delta u_1 + \Delta^2 u_1 = 2\Delta^2 u_1$ by question;

whence $\Delta^2 u_1 = u_2 + \Delta u_1 = 3\Delta u_1 = 3a$,

and $u_3 = 2\Delta^2 u_1 = 6a$.

Proceeding thus, we find $u_4 = 26a$, &c.

The series and the successive orders of differences are thus as follow:—

$$\begin{array}{lllllllll} u = & a, & & 2a, & & 6a, & & 26a \ldots\ldots \\ \Delta = & & a, & & 4a, & & 20a \ldots\ldots & \\ \Delta^2 = & & & 3a, & & 16a \ldots\ldots & & \\ \Delta^3 = & & & & 13a \ldots\ldots & & & \end{array}$$

From which it will be seen that

$$u_2 = 2\Delta^1 u_1,\ u_3 = 2\Delta^2 u_1,\ u_4 = 2\Delta^3 u_1,\ \&c.$$

CHAPTER XXIII.

(230).—The required expression is (Chap. xxii, § [18], formula 3) writing u_x for u_0, and u_{x+n} for u_n,

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{2} u_{x+n-2} - \ldots\ldots$$
$$\pm \frac{n(n-1)}{|\underline{2}} u_{x+2} \mp n u_{x+1} \pm u_x$$

whence $\Delta^6 u_1 = u_7 - 6u_6 + 15u_5 - 20u_4 + 15u_3 - 6u_2 + u_1$

$= 462 - 1512 + 15\kappa - 1120 + 315 - 36 + 1.$

But $\Delta^6 u_1$, by hypothesis $=0$;

therefore $15\kappa = 1890$, and $\kappa = 126$.

The sum of the first ten terms of the series can be readily ascertain by the formula (Chap. xxiv, § [17])

$$\Sigma_1^{10} u_x = 10u_1 + \frac{10.9}{|\underline{2}} \Delta u_1 + \frac{10.9.8}{|\underline{3}} \Delta^2 u_1 + \ldots\ldots$$

whence, inserting the values of u_1, Δu_1, $\Delta^2 u_1$ . . . . ., deduced from the given series, we have

$$\Sigma_1^{10} u_x = 5005.$$

This result could also be arrived at by expanding the series to ten terms and summing the results: thus

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$	$\Delta^5 u_x$	$\Delta^6 u_x$
1	1						
		5					
2	6		10				
		15		10			
3	21		20		5		
		35		15		1	
4	56		35		6		0
		70		21		1	
5	126		56		7		0
		126		28		1	
6	252		84		8		0
		210		36		1	
7	462		120		0		0
		330		45		1	
8	792		165		10		
		495		55			
9	1,287		220				
		715					
10	2,002						
$\Sigma_1^{10} u_x =$	5,005						

(231).—We have, by Maclaurin's theorem (Chap, xxii, § [29]),

$$u_{2\frac{1}{2}} = u_0 + 2\tfrac{1}{2}\left(\frac{d}{dx}\right)u_0 + \frac{(2\frac{1}{2})^2}{\underline{|2}}\left(\frac{d}{dx}\right)^2 u_0 + \ldots\ldots$$

Similarly,

$$u_0 = u_{2\frac{1}{2}} - 2\tfrac{1}{2}\left(\frac{d}{dx}\right)u_{2\frac{1}{2}} + \frac{(2\frac{1}{2})^2}{\underline{|2}}\left(\frac{d}{dx}\right)^2 u_{2\frac{1}{2}} - \,.$$

$$u_5 = u_{2\frac{1}{2}} + 2\tfrac{1}{2}\left(\frac{d}{dx}\right)u_{2\frac{1}{2}} + \frac{(2\frac{1}{2})^2}{\underline{|2}}\left(\frac{d}{dx}\right)^2 u_{2\frac{1}{2}} + \ldots\ldots$$

Adding these two expressions together, the terms involving differential coefficients of the odd orders vanish, and we have

$$u_0 + u_5 = 2u_{2\frac{1}{2}} + (2\tfrac{1}{2})^2\left(\frac{d}{dx}\right)^2 u_{2\frac{1}{2}} + \frac{(2\frac{1}{2})^4}{12}\left(\frac{d}{dx}\right)^4 u_{2\frac{1}{2}} + \ldots\ldots = a$$

Similarly,

$$u_1 + u_4 = 2u_{2\frac{1}{2}} + (1\tfrac{1}{2})^2\left(\frac{d}{dx}\right)^2 u_{2\frac{1}{2}} + \frac{(1\frac{1}{2})^4}{12}\left(\frac{d}{dx}\right)^4 u_{2\frac{1}{2}} + \ldots\ldots = b$$

$$u_2 + u_3 = 2u_{2\frac{1}{2}} + (\tfrac{1}{2})^2\left(\frac{d}{dx}\right)^2 u_{2\frac{1}{2}} + \frac{(\frac{1}{2})^4}{12}\left(\frac{d}{dx}\right)^4 u_{2\frac{1}{2}} + \ldots\ldots = c$$

which becomes

$$a=2u_{2\frac{1}{2}}+\frac{25}{4}\left(\frac{d}{dx}\right)^2u_{2\frac{1}{2}}+\frac{625}{192}\left(\frac{d}{dx}\right)^4u_{2\frac{1}{2}}$$

$$b=2u_{2\frac{1}{2}}+\frac{9}{4}\left(\frac{d}{dx}\right)^2u_{2\frac{1}{2}}+\frac{81}{192}\left(\frac{d}{dx}\right)^4u_{2\frac{1}{2}}$$

$$c=2u_{2\frac{1}{2}}+\frac{1}{4}\left(\frac{d}{dx}\right)^2u_{2\frac{1}{2}}+\frac{1}{192}\left(\frac{d}{dx}\right)^4u_{2\frac{1}{2}}$$

whence
$$\left(\frac{d}{dx}\right)^2u_{2\frac{1}{2}}=-\frac{39(c-b)+5(a-c)}{48}$$

$$\left(\frac{d}{dx}\right)^4u_{2\frac{1}{2}}=\frac{3(c-b)+(a-c)}{2},$$

and inserting these values in the above formula for c, we obtain

$$u_{2\frac{1}{2}}=\frac{c}{2}+\frac{25(c-b)+3(a-c)}{256}$$

the expression given in the question.

(232).—Sunderland's *Notes on Finite Differences* (Chap. ii, § 7).

From the values given in the question, we have

$$\begin{array}{lll} u_4=98{,}391 & & \\ & \Delta u_4=-380 & \\ u_5=98{,}011 & & \Delta^2u_4=-16 \\ & \Delta u_5=-396 & \\ u_6=97{,}615 & & \end{array}$$

Then from the formula

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{|\underline{2}}\Delta^2u_x+\frac{n(n-1)(n-2)}{|\underline{3}}\Delta^3u_x+\&c.$$

we have, making $x=4$, $n=-4$,

$$u_0=u_4-4\Delta u_4+10\Delta^2u_4-20\Delta^3u_4;$$

or inserting the value of u_0 given in the question and the other values as deduced above,

$$100{,}000=98{,}391+1{,}520-160-20\Delta^3u_4;$$

whence
$$\Delta^3u_4=-12{\cdot}45.$$

Then by the formulas

$$u_{10}=u_4+6\Delta u_4+15\Delta^2u_4+20\Delta^3u_4=95,622$$

$$\Delta\ u_{10}=\Delta\ u_4+6\Delta^2u_4+15\Delta^3u_4=-662{\cdot}75$$

$$\Delta^2u_{10}=\Delta^2u_4+6\Delta^3u_4\qquad=-\ 90{\cdot}70$$

$$\Delta^3u_{10}=\Delta^3u_4\qquad=-\ 12{\cdot}45$$

the values of u_{10} to u_{15} can be readily deduced in a tabular form, as follows:—

	Δ^2	Δ	u_x	x
−12·45	− 90·70	− 662·75	95,622·00	10
	−103·15	− 753·45	94,959·25	11
	−115·60	− 856·60	94,205·80	12
	−128·05	− 972·20	93,349·20	13
		−1,100·25	92,377·00	14
			91,276·75	15

(233).—§§ [9], [12]. Sunderland's *Notes on Finite Differences* (Chap. ii, § 7).

(234).—§ [8]. Let the values at 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, and 6 per-cent be represented by u_0, u_1, u_2, u_3, u_4, u_5, and u_6. Then to find u_5, (α) two values u_6 and u_4 being given, we have

$$\Delta u_4=\tfrac{1}{2}(u_6-u_4)=-{\cdot}000819$$

and $$u_5=u_4+\Delta_4={\cdot}020660.$$

(β) Using four values u_2, u_3, u_4, and u_6, we must assume $\Delta^4u_2=0$; then

$$u_2-4u_3+6u_4-4u_5+u_6=\Delta^4u_2=0;$$

whence $$u_5=\frac{u_2+6u_4+u_6-4u_3}{4}={\cdot}0206255$$

(γ).—Using six values, and assuming $\Delta^6u_0=0$,

$$u_6-6u_5+15u_4-20u_3+15u_2-6u_1+u_0=\Delta^6u_0=0;$$

whence $$u_5=\frac{u_6+15u_4+15u_2+u_0-20u_3-6u_1}{6}$$

$$={\cdot}020628.$$

(235).—Let the terms given be u_0, u_2, u_4, u_5; required u_3. As there are four given terms, we must assume $\Delta^4 u_x = 0$;

then $$\Delta^4 u_0 = u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0;$$

also $$\Delta^4 u_1 = u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0.$$

To eliminate u_1, multiply the last equation by 4, and add the result to the first expression. Then

$$4u_5 - 15u_4 + 20u_3 - 10u_2 + u_0 = 0;$$

whence $$u_3 = \frac{15u_4 + 10u_2 - 4u_5 - u_0}{20} = 1 \cdot 7242759.$$

(236).—We have

$$u_5 \;\; = 55; \qquad \Delta u_5 = 71;$$

$$\Delta^2 u_5 = 62; \qquad \Delta^3 u_5 = 30; \qquad \Delta^4 u_5 = 6.$$

Then $$u_2 = u_5 - 3\Delta u_5 + 6\Delta^2 u_5 - 10\Delta^3 u_5 + 15\Delta^4 u_5$$

$$= 4$$

$$u_{12} = u_5 + 7\Delta u_5 + 21\Delta^2 u_5 + 35\Delta^3 u_5 + 35\Delta^4 u_5$$

$$= 3{,}114.$$

These values may be readily proved by extending the series from u_2 to u_{12}, applying the given differences, thus:—

$\Delta^4 u_x$	$\Delta^3 u_x$	$\Delta^2 u_x$	Δu_x	u_x	x
6	12	8	5	4	2
6	18	20	13	9	3
6	24	38	33	22	4
6	30	62	71	55	5
6	36	92	133	126	6
6	42	128	225	259	7
6	48	170	353	484	8
	54	218	523	837	9
		272	741	1,360	10
			1,013	2,101	11
				3,114	12

As an exercise the student may apply Lagrange's theorem, §§ [10]–[12], to ascertain the desired values. We have

$$\begin{array}{lll} u_5 = 55 & a=5 & x=2 \\ u_6 = 126 & b=6 & \\ u_7 = 259 & c=7 & \\ u_8 = 484 & d=8 & \\ u_9 = 837 & e=9 & \end{array}$$

Then, by formula (3),

$$u_2 = \frac{(-4)(-5)(-6)(-7)}{(-1)(-2)(-3)(-4)}\cdot 55 + \frac{(-3)(-5)(-6)(-7)}{1.(-1)(-2)(-3)}\cdot 126$$

$$+\frac{(-3)(-4)(-6)(-7)}{2.1(-1)(-2)}\cdot 259 + \frac{(-3)(-4)(-5)(-7)}{3.2.1(-1)}\cdot 484$$

$$+\frac{(-3)(-4)(-5)(-6)}{4.3.2.1}\cdot 837.$$

$$=(55\times 35)-(126\times 105)+(259\times 126)-(484\times 70)+(837\times 15)$$

$$=47{,}114-47{,}110=4.$$

In a similar manner, putting $x=12$, the value of u_{12} may be ascertained.

(237).—From the values given in the question, we have

$$\begin{array}{ll} u_{235} = 2{\cdot}3710679 & \Delta u_{235} = {\cdot}0018441 \\ \Delta^2 u_{235} = -{\cdot}0000078 & \Delta^3 u_{235} = {\cdot}0000002. \end{array}$$

Then $$u_{235.63} = u_{235} + {\cdot}63\Delta u_{235} + \frac{{\cdot}63({\cdot}63-1)}{\underline{|2}}\Delta^2 u_{235}$$

$$+\frac{{\cdot}63({\cdot}63-1)({\cdot}63-2)}{\underline{|3}}\Delta^3 u_{235}$$

$$= u_{235} + {\cdot}63\Delta u_{235} - {\cdot}117\Delta^2 u_{235} + {\cdot}053\Delta^3 u_{235}$$

$$= 2{\cdot}3722306.$$

(238).—Let

$$\begin{array}{lllll}
 & \Delta & \Delta^2 & \Delta^3 & \Delta^4 \\
a_{21} = u_0 = 18{\cdot}4708 & & & & \\
 & -{\cdot}6564 & & & \\
a_{25} = u_1 = 17{\cdot}8144 & & -{\cdot}0510 & & \\
 & -{\cdot}7074 & & -{\cdot}0054 & \\
a_{29} = u_2 = 17{\cdot}1670 & & -{\cdot}0564 & & -{\cdot}0022 \\
 & -{\cdot}7638 & & -{\cdot}0076 & \\
a_{33} = u_3 = 16{\cdot}3432 & & -{\cdot}0640 & & \\
 & -{\cdot}8278 & & & \\
a_{37} = u_4 = 15{\cdot}5154 & & & &
\end{array}$$

Then
$$a_{30} = u_{2\frac{1}{4}} = u_0 + \frac{9}{4}\Delta u_0 + \frac{9.5}{2\cdot 16}\Delta^2 u_0$$
$$+ \frac{9.5.1}{6\cdot 64}\Delta^3 u_0 + \frac{9.5.1(-3)}{24\cdot 256}\Delta^4 u_0$$
$$= 18{\cdot}4708 - \frac{9}{4}({\cdot}6564) - \frac{45}{32}({\cdot}0510) - \frac{15}{128}({\cdot}0054)$$
$$+ \frac{45}{2048}({\cdot}0022)$$
$$= 16{\cdot}9216.$$

(239).—(*a*) §§ [15]–[25]. (*β*) Assuming fifth differences to be constant, the series of values from u_{x+1} to u_{x+10} can readily be obtained by algebraic addition of the successive differences.

(240).—By the terms of the question, we have

$$\Delta u_x = (\delta u_x + \delta u_{x+1} + \delta u_{x+2} + \ldots + \delta u_{x+n-1})$$
$$= \delta u_x + (\delta u_x + \delta^2 u_x) + (\delta u_x + \delta^2 u_x + \delta^2 u_{x+1})$$
$$+ \ldots + (\delta u_x + \delta^2 u_x + \delta^2 u_{x+1} + \ldots + \delta^2 u_{x+n-2}).$$

But by hypothesis $\delta^2 u_{x+t} = q^t \delta^2 u_x$, $\therefore$ we have

$$\Delta u_x = n\delta u_x + \delta^2 u_x [1 + (1+q) + (1+q+q^2) + \ldots$$
$$+ (1+q+q^2+\ldots q^{n-2})]$$
$$= n\delta u_x + \frac{\delta^2 u_x}{(q-1)^2}[(q^n-1) - n(q-1)],$$

whence,
$$\frac{\Delta u_x - n\delta u_x}{(q^n-1)-n(q-1)} = \frac{\delta^2 u_x}{(q-1)^2}.$$

(241).—We have

$$\begin{array}{ll}
u_6 = 15{\cdot}863 & \\
 & \Delta u_6 = -{\cdot}922 \\
u_7 = 14{\cdot}941 & \\
 & \Delta^2 u_6 = {\cdot}086 \\
u_8 = 14{\cdot}105 & \\
 & \Delta^3 u_6 = -{\cdot}012 \\
u_9 = 13{\cdot}343 & \\
 & \Delta^4 u_6 = {\cdot}005 \\
u_{10} = 12{\cdot}648 &
\end{array}$$

where $u_n =$ the annuity-value at $\frac{n}{2}$ per-cent. Then the value at 4·328 per-cent $= u_{8\cdot656}$, and

$$u_{8\cdot656} = u_6 + 2\cdot656\,\Delta u_6 + \frac{2\cdot656 \times 1\cdot656}{\underline{|2}}\,\Delta^2 u_6$$

$$+ \frac{2\cdot656 \times 1\cdot656 \times \cdot656}{\underline{|3}}\,\Delta^3 u_6 + \frac{2\cdot656 \times 1\cdot656 \times \cdot656 \times (-\cdot344)}{\underline{|4}}\,\Delta^4 u_6$$

$$= u_6 + 2\cdot656\,\Delta u_6 + 2\cdot19[illegible]\,\Delta^2 u_6 + \cdot4809\,\Delta^3 u_6 - \cdot0414\,\Delta^4 u_6$$

$$= 13\cdot5978.$$

Applying the method of central differences, §§ [26]–[32], we have the following values:—

u	Δ	Δ^2	Δ^3	Δ^4
15·863				
	−·922			
14·941		+·086		
	−·836		−·012	
$u_0=$14·105		$b_0=+$·074		$d_0=$·005
	$a_{+1}=-$·762		$c_{+1}=-$·007	
13·343		+·067		
	−·695			
12·648				

and by formula (10),

$$u_x = u_0 + x a_{+1} + \frac{x(x-1)}{\underline{|2}}\,b_0 + \frac{(x+1)x(x-1)}{\underline{|3}}\,c_{+1}$$

$$+ \frac{(x+1)x(x-1)(x-2)}{\underline{|4}}\,d_0,$$

or, inserting the values of $x (= \cdot656)$ and of the successive differences,

$$u_{\cdot656} = 14\cdot105 - (\cdot656 \times \cdot762) - \frac{\cdot656 \times \cdot344}{2}\,(\cdot074)$$

$$+ \frac{1\cdot656 \times \cdot656 \times \cdot344}{6}\,(\cdot007) + \frac{1\cdot656 \times \cdot656 \times \cdot344 \times 1\cdot344}{24}\,(\cdot005)$$

whence

$$u_{\cdot656} = 13\cdot59732.$$

(242).—§§ [39]–[46].

(243).—§§ [40]–[42].

(244).—Since

$$\mu_x = -\frac{1}{l_x}\cdot\frac{d}{dx}\cdot l_x \text{ and } \delta = -\frac{1}{v^x}\cdot\frac{d}{dx}\cdot v^x$$

we have $(\mu_x+\delta) = -\frac{1}{l_x v^x}\cdot\frac{d}{dx}(l_x v^x)$

$$= -\frac{1}{D_x}\cdot\frac{d}{dx}\cdot D_x.$$

But by § [47], formula (25)

$$\frac{du_x}{dx} = \tfrac{1}{2}(u_{x+1}-u_{x-1}) \text{ approximately}$$

$$\therefore \quad \frac{d}{dx}\cdot D_x = \tfrac{1}{2}(D_{x+1}-D_{x-1}) \text{ approximately,}$$

and $(\mu_x+\delta) = -\frac{1}{D_x}\cdot\frac{d}{dx}\cdot D_x = \frac{D_{x-1}-D_{x+1}}{2D_x}$ approximately.

CHAPTER XXIV.

(245).—We have, by the definition of a differential coefficient,

$$\frac{d}{dx}u_x \quad = \text{limit of } \frac{u_{x+h}-u_x}{h} \quad \text{when } h \text{ is infinitely small.}$$

Similarly,

$$\frac{d}{dx}u_{x+h} = \text{,, ,, } \frac{u_{x+2h}-u_{x+h}}{h} \quad \text{,, ,, ,,}$$

$$\frac{d}{dx}u_{x+2h} = \text{,, ,, } \frac{u_{x+3h}-u_{x+2h}}{h} \quad \text{,, ,, ,,}$$

. . . .

$$\frac{d}{dx}u_{x+n-h} = \text{,, ,, } \frac{u_{x+n}-u_{x+n-h}}{h} \quad \text{,, ,, ,,}$$

The sum of the right-hand terms is clearly $= \frac{u_{x+n}-u_x}{h}$; therefore we have

$$\int_x^{x+n}\frac{d}{dx}u_x\,.\,dx = u_{x+n}-u_x$$

$$= \text{Limit of } h\left\{\frac{d}{dx}u_x + \frac{d}{dx}u_{x+h} + \frac{d}{dx}u_{x+2h} + \ldots + \frac{d}{dx}u_{x+n-h}\right\}$$

that is, the integration is equivalent to the sum of an infinite number of infinitely small terms.

(246).—§§ [3], [6].

(247).—Since $u_{x+n}=(1+\Delta)^n u_x$ (Chap. xxii, § [20])

and $\Sigma u_{x+n}=\Delta^{-1}u_{x+n}$ (Chap. xxiv, § [2])

$$\Sigma u_{x+n}=\Delta^{-1}(1+\Delta)^n u_x.$$

Expanding $(1+\Delta)^n$, we have

$$\Sigma u_{x+n}=\Delta^{-1}u_x\left\{1+n\Delta+\frac{n(n-1)}{\underline{|2}}\Delta^2+\ldots\ldots\right\}$$

$$=\Delta^{-1}u_x+n\Delta.\Delta^{-1}u_x+\frac{n(n-1)}{\underline{|2}}\Delta^2.\Delta^{-1}u_x+\ldots\ldots$$

$$=\Sigma u_x+nu_x+\frac{n(n-1)}{\underline{|2}}\Delta u_x+\ldots\ldots$$

and

$$\Sigma u_{x+n}-\Sigma u_x=S_n u_x=nu_x+\frac{n(n-1)}{\underline{|2}}\Delta u_x+\ldots\ldots$$

(248).—Macdonald's *Calculus of Finite Differences* (*Trans. Act. Soc. Edin.*, 1876), page 12.

We have

$$u_1=1c_0+1c_1+1c_2+\ 1c_3+\ 1c_4+\ldots\ldots$$

$$u_2=1c_0+2c_1+4c_2+\ 8c_3+16c_4+\ldots\ldots$$

$$u_3=1c_0+3c_1+9c_2+27c_3+81c_4+\ldots\ldots$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$u_n=1c_0+nc_1+n^2c_2+\ n^3c_3+n^4c_4+\ldots\ldots$$

Summing perpendicularly we have

$$S_n u_x=S_n(x^0)c_0+S_n(x^1)c_1+S_n(x^2)c_2+S_n(x^3)c_3+S_n(x^4)c_4+\ldots\ldots$$

and inserting the values of $S_n(x^0)$, $S_n(x^1)$, &c. . . . . .

$$S_n u_x=nc_0+\frac{n(n+1)}{2}c_1+\frac{n(n+1)(2n+1)}{6}c_2$$

$$+\frac{n^2(n+1)^2}{4}c_3+\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}c_4+\ldots$$

(249).—§ [20]. Sunderland's *Notes on Finite Differences*, Exercise (22).

(250).—From the formula deduced in the solution to Ex. (248)

$$S_n u_x = n c_0 + \frac{n(n+1)}{2} c_1 + \frac{n(n+1)(2n+1)}{6} c_2 + \frac{n^2(n+1)^2}{4} c_3 ;$$

we have, giving to n the successive values, 5, 10, 15, and 20,

$$\begin{aligned}
S_5 u_1 &= 5c_0 + 15c_1 + 55c_2 + 225c_3 = 1{,}365 \\
S_{10} u_1 &= 10c_0 + 55c_1 + 385c_2 + 3{,}025c_3 = 5{,}155 \\
S_{15} u_1 &= 15c_0 + 120c_1 + 1{,}240c_2 + 14{,}400c_3 = 13{,}370 \\
S_{20} u_1 &= 20c_0 + 210c_1 + 2{,}870c_2 + 44{,}100c_3 = 28{,}635
\end{aligned}$$

from which four equations the values of the four unknown quantities, c_0, c_1, c_2, c_3, can be deduced by successive subtraction: thus

$$\begin{aligned}
5c_0 + 15c_1 + 55c_2 + 225c_3 &= 1{,}365 \\
5c_0 + 40c_1 + 330c_2 + 2{,}800c_3 &= 3{,}790 \\
5c_0 + 65c_1 + 855c_2 + 11{,}375c_3 &= 8{,}215 \\
5c_0 + 90c_1 + 1{,}630c_2 + 29{,}700c_3 &= 15{,}265 \\
\hline
25c_1 + 275c_2 + 2{,}575c_3 &= 2{,}425 \\
25c_1 + 525c_2 + 8{,}575c_3 &= 4{,}425 \\
25c_1 + 775c_2 + 18{,}325c_3 &= 7{,}050 \\
\hline
250c_2 + 6{,}000c_3 &= 2{,}000 \\
250c_2 + 9{,}750c_3 &= 2{,}625 \\
\hline
3{,}750c_3 &= 625
\end{aligned}$$

whence

$$c_3 = \frac{5}{30}$$

$$c_2 = \frac{120}{30}$$

$$c_1 = \frac{1{,}075}{30}$$

$$c_0 = \frac{3{,}420}{30}$$

and

$$u_x = \frac{5x^3 + 120x^2 + 1{,}075x + 3{,}420}{30}$$

Macdonald's *Calculus of Finite Differences* (Example xiii).

(251).—(*a*) $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\frac{d}{dx}\log_e l_x$

$$\therefore\quad \log_e l_x = -\int_0^x \mu_x dx + \text{C}$$

and $$l_x = e^{-\int_0^x \mu_x dx + \text{C}} = \kappa e^{-\int_0^x \mu_x dx}$$

where $\kappa = e^{\text{C}}$, and is obviously $= l_0$, the arbitrary value called the radix, of the mortality table. If, as in Makeham's formula, μ_x is in the form $\text{A} + \text{B}c^x$, then

$$l_x = e^{\text{C} - \int_0^x (\text{A}+\text{B}c^x)dx}$$

$$= e^{\text{C} - \text{A}x - \frac{\text{B}c^x}{\log_e c} + \frac{\text{B}}{\log_e c}}$$

which, since e^{C} and $e^{\frac{\text{B}}{\log_e c}}$ are constants, is in the form

$$l_x = ks^x(g)^{c^x}.$$

See Chap. vi, § [15].

(β) When d_x is constant for all ages, μ_x is known to be equal to $\frac{d_x}{l_x} = q_x$, but the expression given in the question would in this case become $\frac{14d_x}{12l_x}$; hence it is clear that the coefficients, either in the numerator or denominator, are in error. The correct expression is

$$\mu_x = \frac{7(d_{x-1}+d_x) - (d_{x-2}+d_{x+1})}{12l_x}.$$

(252).—§§ [21]–[26].

(253).—§ [82]. Σu_x may be written symbolically as $\Delta^{-1}u_x$ (see § [2]), and $\Delta^{-1}u_x = [(1+\Delta)-1]^{-1}u_x = (e^{\frac{d}{dx}}-1)^{-1}u_x$.

The expression in brackets may obviously be expanded in a series of terms involving the powers of the symbol $\frac{d}{dx}$, which, when combined with the symbol of quantity, u_x, become the successive differential coefficients of u_x.

Assume that, for all values of $\frac{d}{dx}$,

$$(e^{\frac{d}{dx}}-1)^{-1}=A\left(\frac{d}{dx}\right)^{-1}+B+C\left(\frac{d}{dx}\right)+D\left(\frac{d}{dx}\right)^2+\ldots\ldots\quad\text{(i)}$$

then

$$(e^{-\frac{d}{dx}}-1)^{-1}=-A\left(\frac{d}{dx}\right)^{-1}+B-C\left(\frac{d}{dx}\right)+D\left(\frac{d}{dx}\right)^2-\ldots$$

and

$$(e^{\frac{d}{dx}}-1)^{-1}+(e^{-\frac{d}{dx}}-1)^{-1}=B+D\left(\frac{d}{dx}\right)^2+F\left(\frac{d}{dx}\right)^4+\ldots\ldots\quad\text{(ii)}$$

involving only the constant B, and even powers of $\frac{d}{dx}$.

But $$(e^{\frac{d}{dx}}-1)^{-1}+(e^{-\frac{d}{dx}}-1)^{-1}=(e^{\frac{d}{dx}}-1)^{-1}+e^{\frac{d}{dx}}(1-e^{\frac{d}{dx}})^{-1}$$
$$=(e^{\frac{d}{dx}}-1)^{-1}(1-e^{\frac{d}{dx}})$$
$$=\frac{1-e^{\frac{d}{dx}}}{e^{\frac{d}{dx}}-1}=-1$$

hence, in the series (ii), $B=-1$, and the coefficients of the remaining terms (D, F, &c.) all $=0$.

(254).—§ [39].

(255).—(α) Formula (24), § [34].
(β) Formula (25), § [36].
(γ) Formula (26), § [37].

(256).—§ [40].

(257).—§§ [42]-[55]. *J.I.A.*, xxiv, 95. Owing to the simplicity of the coefficients in formulas (33) and (36), they can be readily applied to numerical calculation, and will generally be found to give accurate results.

(258).—Chap. ix, § [19].

(259).—Chap. ix, §§ [21], [22], [25].

(260).—Chap. xii, §§ [58], [61], [62].

(261).—Chap. xiii, §§ [51]-[58].

(262).—Chap. xiv, § [41], formula (27).

(263).—Chap. xv, § [11].

(264).—Chap. xv, § [20], [25].

(265).—We have

$$\bar{A}^{1}_{x:\overline{y:z}:(3)}={}_{t|}\bar{A}_x+\int v^{t+n}\frac{1}{l_x}\left(\frac{l_{y+n}}{l_y}+\frac{l_{z+n}}{l_z}-\frac{l_{y+n}l_{z+n}}{l_yl_z}\right)l_{x+t+n}\mu_{x+t+n}\cdot dn.$$

The value of the co[illegible] continuous annuity is

$$\bar{a}_{x|\overline{y.z}(\bar{t})} = {}_{t|}\bar{a}_x + \int_0^t v^{t+n}\frac{1}{l_x}\left(\frac{l_{y+n}}{l_y}+\frac{l_{z+n}}{l_z}-\frac{l_{y+n}l_{z+n}}{l_yl_z}\right)l_{x+t+n}\cdot dn.$$

The values of the continuous annuity, and of the single premium for the corresponding assurance, can be obtained in one operation by the use of one of the formulas of approximate summation given in Chap. xxiv. The divisor for the annual premium (payable until the expiration of the risk) may be deduced from the above continuous annuity by the formula

$$1+a_{x|\overline{y.z}.(\bar{t})} = \bar{a}_{x|\overline{y.z}(\bar{t})}+\frac{1}{2}+\frac{u_x+\delta}{12}.$$

[illegible]: C. & E. Layton, London.